1. LAGRANGIAN FLOER HOMOLOGY (CONTD)

Recall first our approaches to $CF^*(L, L)$ with the $A_\infty$ algebraic structure:

1. Hamiltonian perturbations $CF^*(L, L) = \Lambda^{[L;\varphi_H(L)]}$
2. FOOO: $CF^*(L, L) = C_*(L, \Lambda)$ the space of “chains” on $L$. We have evaluation maps $ev_i: \overline{\mathcal{M}}_{0,k+1}(M, L; J, \beta) \to L$, giving multiplication maps $m_k(C_k, \ldots, C_1) = \sum_{\beta \in \pi_2(X, L)} T^{\omega(\beta)}(ev_0)_*[\overline{\mathcal{M}}_{0,k+1}(M, L; J, \beta)] \cap ev_1^*C_1 \cap \cdots \cap ev_k^*C_k$

3. Cornea-Lalonde approach: “clusters”. Pick a Morse function $f: L \to \mathbb{R}$, and set $CF^*(L, L) = \Lambda^{\text{crit}(f)}$. $m_k$ counts “clusters” of $J$-holomorphic disks and gradient flowlines.

1.1. Disks and Obstruction. We’ve seen that, if $L_0$ or $L_1$ bound holomorphic disks, then $\partial^2 \neq 0$ (the moduli space of index 2 strips has disk bubbling on the boundaries in addition to strips). Counting the contribution of disk bubbles gives $m_0 \in CF^*(L, L)$. In FOOO theory, $m_0 = \sum_{\beta \neq 0} ev_*[\overline{\mathcal{M}}_{0,1}(M, L; J, \beta)] \cdot T^{\omega(\beta)}$. A bubble on the boundary of the disk on $L_1$ is $m_2(m_0, p)$, for $p \in CF^*(L_0, L_1)$, $m_0 \in CF^*(L_1, L_1)$. Hence $m_0$ is the obstruction to $\partial^2 = 0$. More generally, $A_\infty$-equations still hold if we include the terms $m_k(\cdots, m_0, \cdots)$, which we can interpret as disks with $k + 1$ marked points developing disk bubbles on the boundary. This is called a “curved $A_\infty$-category”. We say that $L$ is unobstructed if $m_0 = 0$, and weakly unobstructed if $m_0 \in \Lambda_1$, where $1_L$ is the fundamental chain $[L]$. This implies centrality, and $m_1^2 = 0$ on $CF(L, L)$. Weakly unobstructed $L$’s with a given “charge” form an honest $A_\infty$-category.

In FOOO, one tries to cancel the obstruction by a formal deformation $b \in CF^1(L, L)$. For $\nabla = d + b$ on $CF^*(L, L)$, write

$$m_k^b(C_k, \ldots, C_1) = \sum m_{k+\ell}(b \cdots b, c_k, b \cdots b, \ldots, b \cdots b, c_1, b \cdots b)$$

This is still a curved $A_\infty$-algebra, and we look for $b$, s.t. $m_0^b = m_0 + m_1(b) + m_2(b, b) + \cdots = 0$. Such a $b$ is called a “bounding cochain”. One can similarly define weakly bounding cochains, and define our objects to be equivalence classes of pairs $(L, b)$ for $b$ a weakly bounding cochain.
1.2. **Coherent Sheaves on a Complex Manifold.** Let $X$ be a complex manifold, $\mathcal{O}_X$ the sheaf of holomorphic functions on $X$. Recall that a coherent sheaf $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules s.t.

- $\mathcal{F}$ is of finite type, i.e. there is an open cover by affines $U_i$ s.t. $\mathcal{F}|_{U_i}$ is generated by a finite number of sections, i.e. $\exists$ surjective maps $\mathcal{O}_X|_{U_i} \rightarrow \mathcal{F}|_{U_i}$.
- For all $U \subset X$ open, $\phi : \mathcal{O}_X|_{U} \rightarrow \mathcal{F}|_{U}$ a homomorphism of $\mathcal{O}_X$-module, Ker $\phi$ is of finite type.

If $X$ is nice enough, $\mathcal{F}$ has finite presentation, i.e. $\exists$ an open cover s.t. there is an exact sequence

$$0 \rightarrow \mathcal{O}_X^{\oplus r} \rightarrow \mathcal{O}_X^{\oplus n} \rightarrow \mathcal{F}|_U \rightarrow 0$$

i.e. a coherent sheaf is the cokernel of a morphism of vector bundles. Coherent sheaves form an abelian category, i.e. they contain kernels and cokernels.

**Example.** Any vector bundle $E$ can be thought of as a locally free sheaf of holomorphic sections. For $D$ a hypersurface defined by $s = 0$ for a section of some line bundle $L$, we have a short exact sequence

$$0 \rightarrow L^{-1} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$$

For $Z \subset X$ a codimension $r$ subvariety defined transversely as $s^{-1}(0)$, for $s$ a section of a rank $r$ vector bundle $\mathcal{E}$, we have a Koszul resolution

$$0 \rightarrow \bigwedge^r \mathcal{E}^* \rightarrow \bigwedge^{r-1} \mathcal{E}^* \rightarrow \cdots \rightarrow \mathcal{E}^* \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

For $X$ smooth (proper?), coherent sheaves always have a finite resolution by vector bundles.

The category of sheaves has both an internal $\mathcal{H}$ (which is a sheaf) and an external Hom (just a group, and in fact the global sections for the former). A functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is left exact if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \implies 0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$. If the category $\mathcal{C}$ has enough injectives (objects such that $\text{Hom}_C(\_ , I)$ is exact), there are right-derived functors $R^iF$ s.t.

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow R^1F(A) \rightarrow R^1F(B) \rightarrow R^1F(C) \rightarrow \cdots$$

To compute $R^iF(A)$, resolve $A$ by injective objects as $0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$, we get a complex $0 \rightarrow F(I^0) \rightarrow F(I^1) \rightarrow F(I^2) \rightarrow \cdots$. Taking cohomology gives $R^0F(A) = \text{Ker} (F(I^i) \rightarrow F(I^{i+1}))/\text{im} (F(I^{i-1}) \rightarrow F(I^i))$. Note that $R^0F(A) = F(A)$.

**Example.** Sheaf cohomology arises as the right derived functor of the global section functor, and can be computed by acyclic sheaves (e.g. flasque sheaves).