0.1. Lagrangian Floer Homology (contd). Let \((M, \omega)\) be a symplectic manifold, \(L_0, L_1\) compact Lagrangian submanifolds intersecting transversely. Recall that the complexes \(\text{CF}^*(L_0, L_1) = \Lambda|L_0 \cap L_1|\) carry a differential \(m_1\), product \(m_2\), and higher operations \(\text{CF}^*(L_0, \ldots, L_k) \otimes \cdots \otimes \text{CF}^*(L_{k-1}, L_k) \xrightarrow{m_k} \text{CF}^*(L_0, L_k)[2-k]\) (1)

We looked at \(J\)-holomorphic maps \(u\) from disks \(D^2\) with marked boundary points to disks in the manifold between \(L_0, \ldots, L_k\) with \(u(z_0) = q \in L_0 \cap L_k\), \(u(z_i) = p_i \in L_{i-1} \cap L_i\). We find that the expected dimension of our manifold \(\mathcal{M}(p_1, \ldots, p_k, q, [u], J)\) is \(\deg q - (\deg p_1 + \cdots + \deg p_k) + k - 2\). Assuming transversality,

\[
m_k(p_k, \ldots, p_1) = \sum_{q \in L_0 \cap L_k \atop \text{ind}([u]) = 0} (\# \mathcal{M}(p_1, \ldots, p_k, q, [u], J)) T^\omega(u) q \tag{2}
\]

By looking at the \(\partial\) (1-dimensional moduli space), we obtained the \(A_\infty\) relations:

**Proposition 1.** Assuming no bubbling of disks and spheres, \(\forall m \geq 1, (p_1, \ldots, p_m), p_i \in L_{i-1} \cap L_i\),

\[
\sum_{k, \ell \geq 1 \atop k + \ell = m + 1 \atop 0 \leq j \leq \ell - 1} (-1)^* m_\ell(p_m, \ldots, p_{j+k+1}, m_k(p_{j+k}, \ldots, p_{j+1}), p_j, \ldots, p_1) = 0 \tag{3}
\]

where \(\ast = \deg(p_1) + \cdots + \deg(p_j) + j\).

This implies that \(m_1\) is a differential, \(m_2\) satisfies the Leibniz rule, and \(m_2\) is associative up to homotopy given by \(m_3\) (i.e. it is associative in \(HF^*\)).

**Definition 1.** An \(A_\infty\) category is a linear “category” where morphism spaces are equipped with algebraic operations \((m_k)_{k \geq 1}\) satisfying the \(A_\infty\)-relations (those defined above).

In our case, we have the following categories:
• A Fukaya category is any of a number of $A_\infty$ categories whose objects are Lagrangian submanifolds (with extra data), the morphisms are Floer complexes, and the algebraic operations are as above.
• So far we only have an ‘$A_\infty$-precategory” because the homomorphisms have only been defined for transverse pairs of objects.
• At the homology level, we can also define the Donaldson-(Fukaya category) whose homomorphisms are the cohomologies $HF$, so that composition is automatically associative. This is technically easier, but we lose some information that we need for mirror symmetry.
• We eventually want to define our Fukaya category to be over $\mathbb{C}$, rather than over the Novikov ring. So far, we have counted disks with weights $T^{\omega(u)} \in \Lambda$, and Gromov compactness tells us that there are only finitely many contributions below a certain area. That is, the sums may be infinite, but they converge in the Novikov ring. Physicists usually write the terms as $e^{-2\pi \omega(u)} \in \mathbb{R}$ instead of $T^{\omega(u)}$, and hope for convergence.

For Lagrangians $L_i$ equipped with $(E_i, \nabla_i) \to L_i$ complex vector bundles with flat (unitary) connections. We think of these as local systems of coefficients on our Lagrangians. We define an associated complex with twisted coefficients:

\[
\text{for } L_0, L_1 \text{ transverse. Then given } p_1, \ldots, p_k, p_i \in L_{i-1} \cap L_i, w_1, \ldots, w_k, w_i \in \text{Hom}((E_{i-1})_{p_i}, (E_i)_{p_i}), \text{ we let}
\]

\[
m_k(w_k, \ldots, w_1) = \sum_{q \in L_0 \cap L_k} (\# \mathcal{M}(p_1, \ldots, p_k, q, [u], J)) T^{\omega(u)} \mathcal{P}_{[\partial u]}(w_k, \ldots, w_1)
\]

where $\mathcal{P}_{[\partial u]}(w_k, \ldots, w_1) \in \text{Hom}((E_0)_q, (E_k)_q)$ is defined by

\[
\mathcal{P}_{[\partial u]}(w_k, \ldots, w_1) = \gamma_k \circ w_k \circ \gamma_{k-1} \circ \cdots \circ \gamma_1 \circ w_1 \circ \gamma_0
\]

where parallel transport along $\partial u$ from $q \to p_1$ gives $\gamma_0 \in \text{Hom}((E_0)_q, (E_0)_{p_1})$, and similarly parallel transport from $p_i \to p_{i+1}$ using $\nabla_i$ gives $\gamma_i \in \text{Hom}((E_i)_{p_i}, (E_i)_{p_{i+1}})$. For $\nabla_i$ flat, this only depends on $[\partial u]$. In particular, if $E_i$ is the topologically trivial line bundle $\mathbb{C} \times L_i$ and $\nabla_i$ is a flat $U(1)$
connection (up to gauge equivalence), $\nabla_i = d + iA_i$ for $A_i$ a closed 1-form, this encodes the data of holonomies $\pi_1(L_i) \to U(1)$. Then, using trivializations, we get $CF = \Lambda_{C}^{[L_0 \cap L_1]}$ with generators $p, w = id : E_0 \to E_1$ and $m_k$ counts disks with weight $T^\omega(u) \cdot \operatorname{hol}(\partial u)$, where

$\operatorname{hol}(\partial u) = \exp \left( i \sum_{j=0}^{k} \int_{\partial u_j} A_j \right)$

is the holonomy of parallel transport.

We can now construct our first iteration of the Fukaya category:

- The objects are $\mathcal{L} = (L, E, \nabla)$, where $L$ is a compact spin Lagrangian ($\mathbb{Z}$-graded: $\mu_L = 0$ with grading data) and $(E, \nabla)$ a flat hermitian vector bundle.
- The morphisms for transverse $L_0, L_1$ is given by $\operatorname{hom}(L_0, L_1) = CF^\ast$.

Issues:

1. What if $L_0$ is not transverse to $L_1$ (in particular, if $L_0 = L_1$)?
2. What if $L$ bounds disks?

For the first problem, see Seidel’s book: the idea is to use a Hamiltonian perturbation $\phi_H$ to get $L_1$ to be transverse to $L_0$, and define $CF^\ast(L_0, L_1)$ to be generated by $L_0 \cap \phi_H(L_1)$ (the vector bundles carry without change). We perturb all the $\bar{\partial}$-equations by suitable terms: in the strip-like ends, we have $\frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} + X_{H}(u)) = 0$ for $H = H(L_i - 1, L_i)$. We need a procedure to associate to $(L, L')$ a Hamiltonian $H(L, L')$, and to a sequence $L_0, \ldots, L_k$ some compatible perturbation data, and further to show that different choices give equivalent $A_\infty$-categories. Note that this will not be strictly unital, and will only get a homology unit.

Alternatively, one can use “Morse-Bott” Floer theory (e.g. FOOO). We define $CF^\ast(L, L) = C_\ast(L; \Lambda)$ to be the space of singular chains on $L$: when defining the operations $m_k$, instead of strip-like ends, we have a marked point $z$ on the boundary such that when evaluating at $z$, and require $u(z)$ to be in the chain. For instance, in the product $m_2$, one considers disks with boundary points $z_0, z_1, z_2$ with three evaluation maps $\operatorname{ev}_i : \bar{\mathcal{M}}_{0,3}(M, L; J, \beta) \to L$,

$m_2(C_2, C_1) = \sum_{\beta \in \pi_2(X, L)} T^\omega(\beta)(\operatorname{ev}_0)_*([\bar{\mathcal{M}}_{0,3}(M, L; J, \beta)] \cap \operatorname{ev}_1^* C_1 \cap \operatorname{ev}_2^* C_2)$

For the class $\beta = 0$, we find that the contribution of constant disks gives the intersection product on $C_\ast(L)$. The higher $m_k$ follow similarly, though $m_1$ does not allow $\beta = 0$ and adds the classical $\partial C$ instead. More generally, if $L_0 \cap L_1$ have a “clean intersection” (i.e. $L_0 \cap L_1$ is smooth), then we set $CF^\ast(L_0, L_1) = C_\ast(L_0 \cap L_1; \Lambda)$. 