Lecture 3:

Last time: deformation of $(x, J)$ given by $\{ s \in \mathcal{L}^0(\theta, TX) \} / \mathcal{O} = \{ s \in \Omega^0(\theta, TX) \} / \mathcal{O}$

quotient by $\text{Diff}(x) \to 1^{st} \text{ order: } \text{Diff}(x) = H^1(x, TX)$, but obstruction $\in H^2(x, TX)$

Then (Boigamer - Tian - Todorov)

\[ X \text{ compact Calabi-Yau with } H^0(x, TX) = 0 \Rightarrow \text{deformation } g \text{ of } X \text{ are undisturbed, i.e. } \mathcal{M}(x, g) \text{ is locally smooth w/ tangent space } \cong H^1(x, TX). \]

assuming $\text{Aut}(x, J) = 1$

For CY $n$-folds, $TX \cong \Omega_{x}^{n-1}$ so $H^0(x, TX) = H^{n-1, 0}_0 \cong H^{0, n-1}_0 \leftarrow \text{assume 0}$

$H^1(x, TX) = H^{n-1, 1}_0 \leftarrow \text{deformation}$

$H^2(x, TX) = H^{n-1, 2}_0 \leftarrow \text{obstruction}$

Recall: Hodge theory on compact Kähler manifolds:

- Kähler metric $\leftrightarrow$ operator $a d^* x = -*d^* x$
- Laplacian $\Delta = dd^* + d^* d$, $\Box = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$; $\Delta = 2 \Box$
- Every cohomology class has a unique harmonic representative $\bar{\partial}$-harmonic (Hodge decomposition)

so $H^k_{\bar{\partial}}(x, \mathcal{C}) \cong \ker(\Delta : \mathcal{L}^k(x, \mathcal{C}))$

$= \ker(\Box : \mathcal{L}^k(x, \mathcal{C}))$

$= \bigoplus \ker(\Box : \mathcal{L}^p, q) \cong \bigoplus H^{p, q}_0(X)$

- given $H^{p, q} = H^{n-q, n-p}$, complex conj: $H^{p, q} \cong \bar{H}^{q, p}$

so Hodge diamond $h^{p, q}$ is symmetric.

For a CY $n$-fold, $H^{p, 0}_0 = H^{n-p, n-p}_0 = H^{n-p}(x, \mathcal{O}_x) \cong H^{n-p}(x, TX) = H^{0, n-p}_0$

so $h^{p, 0} = h^{n-p, 0}$

For a CY 3-fold

Under assumption $h^{1, 0} = 0$, get:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & h^{2, 1} & h^{2, 1} & 1 \\
0 & h^{1, 1} & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
Another interpretation of Kodaira-Spencer map for Calabi-Yaus:

\[ X = \mathcal{X} \]

Family of deformation of \((X, J) \rightarrow (X, J_t)_{t \in S}\)

\[ S \geq 0 \]

\[ c_1(K_X) = 0 \quad \text{(deform. int.) and } H^{0,1} = 0 \quad \text{assumed} \]

\( K_X \sim O_X \) holomorphically even after deformation (so \( M(X, J_t) \) a Calabi-Yau)

Then \( \exists [\Omega_t] \in H^{0,0}_X(X) \subseteq H^n(X, \mathbb{C}) \). \( \Omega^n \): how does it depend on \( t \)?

Holomorphic form

\[ \frac{\partial \Omega}{\partial t} \in T_0 S \]

\[ \frac{\partial \Omega}{\partial t} \in \Omega^{n,0} \oplus \Omega^{n-1,1} \] because of

Thm. (Graffites transversality) (proved last time)

\[ \alpha_t \in \Omega^{0,q}(X, J_t) \Rightarrow \frac{\partial \alpha_t}{\partial t} \in \Omega^{0,q} + \Omega^{1,q-1} + \Omega^{1,q+1} \]

Now: \( \frac{\partial \Omega}{\partial t} \) is \( \Omega \)-closed (since \( \Omega \)-closed)

\[ \Rightarrow \left( \frac{\partial \Omega}{\partial t} \right)^{(n-1,1)} \text{ is } \Omega \text{-closed } \Rightarrow \exists \left[ \frac{\partial \Omega}{\partial t} \right]^{(n-1,1)} \in H^{n-1,1}(X) \]

- For fixed \( \Omega_0 \), this is indep. of choice of \( J_t \). Indeed, could rescale to \( f(t) \Omega_t \), but then \( \frac{\partial}{\partial t} (f(t) \Omega_t) = \frac{df}{dt} \Omega_t + f(t) \frac{\partial \Omega_t}{\partial t} \)

\[ \Omega^{(n-1,1)} \text{ part scale linearly} \]

- As seen above, \( H^{n-1,1}(X) = H^1(X, \Omega^{n-1}) \approx H^1(X, TX) \)

The identification \( TX = \Omega^{n-1}_X \) also depends on a choice of \( \Omega \).

The image of \( \frac{\partial \Omega}{\partial t} \) in \( H^1(X, TX) \) is indep. of choice and = Kodaira-Spencer map.

\[ \left[ \frac{\partial \Omega}{\partial t} \right]^{(n-1,1)} \in H^{n-1,1}(X) \]

Hence: for \( \Theta \in H^1(X, TX) \) deform. of complex structure,

\[ \Theta, \Omega \in H^1(X, \Omega^{0,0}_X \oplus TX) \approx H^1(X, \Omega^{n-1}_X) = H^{n-1,1}(X) \]

and \( \left[ \nabla_\Theta \Omega \right]^{(n-1,1)} \in H^{n-1,1}(X) \) are the same thing.

Iterating to 3rd order variation ... on a CY-3 field,

\[ \langle \Theta_1, \Theta_2, \Theta_3 \rangle := \sum_X \nabla \wedge (\Theta_1 \wedge \Theta_2 \wedge \Theta_3 \wedge \Omega) = \sum_X \Omega \wedge \nabla_{\Theta_1} \nabla_{\Theta_2} \nabla_{\Theta_3} \Omega \]

(Need 3 derivatives before we can hit a nontrivial (0,3) component...)
Pseudoholomorphic curves: (Reference: McDuff-Salamon book)

\((X^n, \omega)\) symplectic manifold, \(J\) compatible almost-\(\mathbb{C}\) structure
\((J^2 = -1, \omega(J), J)\) Hermitian metric.

\((E, j)\) Riemann surface of genus \(g\), \(z_1, \ldots, z_k \in E\) marked points.

Moduli space \(M_{g,k} = \{(E, j, z_1, \ldots, z_k)\}/\text{biholomorphism}\) \((\text{dim}_o = 3g-3+k)\)

Main case for \(m_o\): \((S^2, j)\), \(0,1,\infty\) : \(M_{0,3} = \{\text{pt}\}\) so we won't discuss moduli space further.

\(\text{Def.}\ |
\| u: E \to X \text{ is } J\text{-holomorphic if } J \cdot du = du \cdot j \\
\| \text{ ie. } \overline{\partial}_J u = \frac{i}{2} (du + J \cdot du) = 0 \in \Gamma(E, \mathfrak{S}^{0,1}_{\overline{\partial}} u^* TX)\)

\(\text{Def.}\ |
\| M_{g,k} (x, J, \beta) = \{ (E, j, z_1, \ldots, z_k), u: E \to X \}/\sim \| \beta \in H_2 (x) \\
\| (\text{equivalence relation: } \phi: E \cong E', \phi(z_i) = z'_i, \phi \overline{\partial}_J u = 0 \to X)\)

\(\text{ie. zero set of a section } \overline{\partial}_J \circ \mathfrak{S}^{0,1}_{\overline{\partial}} u^* TX = \Gamma(E, \mathfrak{S}^{0,1}_{\overline{\partial}} u^* TX)\)

\(\text{More precisely, look at } W^{k+1, p} \text{ maps, and } E_u = W^{k, p} (E, \mathfrak{S}^{0,1}_{\overline{\partial}} u^* TX)\)

\(\text{\sim Banach bundle over a Banach manifold}\)

The linearized operator \(D_{\overline{\partial}}: W^{k+1, p} (E, u^* TX) \times TM_{g,k} \to W^{k, p} (E, \mathfrak{S}^{0,1}_{\overline{\partial}} u^* TX)\)
\(D_{\overline{\partial}} (v, j') = \overline{\partial}_J v + \frac{i}{2} \nabla_J \cdot du \cdot j + \frac{i}{2} J \cdot du \cdot j'\)

is Fredholm, of index \(r = \text{dim } M_{g,k} - 2d = 2 \langle c_1(TX), \beta \rangle + n(2-2g) + (6g-6+2k)\)

\(Q:\) transversality? \(\text{ie. can we get } D_{\overline{\partial}} \text{ to be onto at } \beta \in M_{g,k} (x, J, \beta)?\)

\(\text{say } u \text{ is regular if } D_{\overline{\partial}} \text{ onto at } u.\)

\(\text{If so then } M_{g,k} (x, J, \beta) \text{ is smooth of dimension } 2d.\)
Def: \( u: \Sigma \to X \) is simple ("somewhere injective") if \( \exists z \in \Sigma \) st. \( \{ du(z) \text{ injective} \} \Rightarrow u^{-1}(u(z)) = \{z\} \).

Then, \( u \) is a covering \( \Sigma \to \Sigma \to X \):

\[ M_{g,k}^*(X,J,\beta) = \{ \text{simple J-hol. curve} \} \]

Thm: \( J^{reg}(X,\beta) = \{ J \in J(X,\omega) / \text{every simple J-hol. curve in class } \beta \text{ is regular} \} \)

is a Baire subset in \( J(X,\omega) \)

For \( J \in J^{reg}(X,\beta) \), \( M_{g,k}^*(X,J,\beta) \) is smooth of real dim. \( 2d \)

and can't have a natural orientation.

\( \triangleright \) in general \( M_{g,k} \text{ orbifold} \) (\( \Sigma \) with automorphisms)

Proof sketch:

- Consider \( D_3 u = 0 \) as equiv. \( \text{Map}(\Sigma, X) \times M_{g,k} \times J(X,\omega)) \Rightarrow (u, J, \beta) \)

then linearization is surjective for all simple maps.

(while fails for multiple covers...)

"univ. moduli space" \( \tilde{M}_{g,k}^* \xrightarrow{\pi_3} J(X,\omega) : \) projection to \( J \) is Fredholm.

\( \circledR \) by Sard-Smale a generic \( J \) is a regular value.

and hence \( M_{g,k}^*(X,J,\beta) \) is smooth.

- Orientation: need over \( D^3 \) on \( \ker(D_3) \).

If \( J \) is integrable

then \( D_3 \) is \( C \)-linear and \( \exists \) natural orientation. Can still do it in general.

* Moreover: \( \forall J_0, J_1 \in J^{reg}(X,\beta) \), \( \exists \) (dense set of choices \( \delta \)) path \( \{ J_t \}_{t \in [0,1]} \) s.t.

\[ \prod_{t \in [0,1]} M_{g,k}^*(X,J_t,\beta) \]

smooth cobordism between \( M_{g,k}^*(X,J_0,\beta) \) and \( M_{g,k}^*(X,J_1,\beta) \)

but we need a compactness result, else "#curve" not index of \( J \in J^{reg}(X) \)!

* So far we haven't used the symplectic form as much... it's used for

Thm: (Gromov compactness)

\( u_n : \Sigma_n \to X \) sequence of J-holomorphic curves, \( J \in J(X,\omega) \),

\( E(u_n) = \int_{\Sigma_n} \omega = c(\omega) \), \( u_n[\Sigma_n] \) bounded \( \Rightarrow \)

\( \exists \) subsequence that converges to a stable map \( u_\infty : \Sigma_\infty \to X \)