Recall: Motivation for SYZ conjecture:

Q: How does one build a mirror $X^!$ of a given Calabi-Yau manifold $X$?

Observations:

- Hori says $\text{D}^b\text{Ch}(X^!) \cong \text{D}^\text{st}\text{Fuk}(X)$

| $p \in X^!$ point | $\Theta_p \in \text{D}^b\text{Ch}(X^!)$ | $L_p \in \text{D}^\text{st}\text{Fuk}(X)$. |

$X^! =$ moduli space of stable sheaves in $\text{D}^b\text{Ch}(X^!)$

$\text{moduli space of certain objects in } \text{D}^\text{st}\text{Fuk}(X)$. 

$\text{HF}^*(L_p, L_p) \cong \text{Ext}^0(\Theta_p, \Theta_p) \cong H^0(\mathcal{T}_p; \mathcal{O}) \Rightarrow$ reasonable guess:

\[ \text{generic points of } X^! \text{ correspond to isomorphism classes of } (L, V), \]

\[ L \subset X \text{ Lagrangian, } V \subset \mathcal{O}(1). \text{ Not Grim}. \]

(some points of $X^!$ might still only correspond to objects of the derived Fukaya category).

The Strominger-Yau-Zaslow conj. (1996) builds on this and gives a richer geometric picture (get both ex. & sympl. geometry on each of $X, X^!$) by picking a preferred representative of the isom. class of $(L, V)$ (doesn't always exist $\triangle$).

**SYZ conj:** $X, X^!$ carry dual fibrations by special Lagrangian tori.

\[ \text{I.e.: } \mathcal{T}^n \rightarrow X, \quad \mathcal{T}^n \rightarrow X^! \quad \downarrow \pi \quad \downarrow \pi^! \]

\[ \text{where } \pi^! = \text{Hom}(\pi_1 \mathcal{T}, \mathcal{O}(1)) \quad \text{dual torus} \]

\[ \text{i.e. } \check{X} = \{ (L, V) \mid L \text{ fiber of } \pi, \quad V \in \text{hom}(\pi_1 L, \mathcal{O}(1)) \}\] & vice-versa.

**Special Lagrangian:** $\omega_{1L} = 0$ and $\text{Im}(\pi_2)_{L_0} = 0$ in holomorphic volume form.

We'll look more into it but there are several warnings:

- Contracting sing from fibrations is difficult & usually impossible.
  (Joyce, Haase-Zabhak, Gross-Siebert, ...)
general slogan: A LCSL degeneration should give rise to a SLAG fibration (the CY metric collapses to $B$). Still very hard.

(Also note: different choice of LCSL degeneration should give a different SLAG fibration, and hence a different mirror).

* SLAG fibration will usually have singularities $\Rightarrow$ dual fibration not well-defined. A related issue = “instanton corrections”

So conjecture as stated mostly applies to tori... needs to be adjusted in general.

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**Special Lagrangian submanifolds:**

- $X, \omega, j$ Kähler, $g$ Kähler metric, $\Omega \in \Omega^{n,0}$ holom. volume form
- strict Calabi-Yau: $g$ Ricci-flat, $|\Omega|_g$ const. $\Leftrightarrow$ almost CY: $|\Omega|_g = \psi \in C^\infty(X, \mathbb{R})$

- point: curvature of Chern $\Omega \wedge \bar{\Omega} = c(n) \omega^n$
- connection on $\Omega^{n,0} \equiv$ Ricci form; strict CY $\Rightarrow$ $\Omega^{n,0} = 0$

Fact: $L \subset X$. Lagrangian submanifold $\Rightarrow \Omega_{1L} \in \Omega^n(L, \mathbb{C})$ is of the form

$$\Omega_{1L} = e^{i\eta} \psi \operatorname{vol}_{g_{1L}}$$

with $e^{i\eta}: L \to S^1$ phase function.

(Pf: linear algebra! at a point $p \in L$, $\exists$ basis of $T_p X$ s.t.

$$(T_p X, \omega_p, J_p, T_p L) \cong (\mathbb{C}^n, \omega_0, J_0, \mathbb{R}^n),$$

and $\Omega_p = e^{i\eta(p)} \psi(p) \operatorname{vol} d\eta$)

**Def:** $L$ is special Lagrangian if the phase function is constant.

Then $\int_L \Omega \in e^{i\eta} \mathbb{R}_+$. Given $[L] \in H_n(X, \mathbb{Z})$ normalize $\Omega$ so that $\int_L \Omega = 1$.

$\Rightarrow$ **Def.** $L$ is special Lagrangian iff $\operatorname{Im} \Omega_{1L} = 0$.

(and then $\operatorname{Re} \Omega_{1L} = \psi \cdot \operatorname{vol}_L$, up to suitable choice of orient of $L$)

**Rank 1:** in strict CY case, special Lagrangians are calibrated & hence volume-minimizing in their homology class: $\operatorname{Re} \Omega_{1TT} \leq \operatorname{vol}_{g_{1TT}}$ $\forall$ TT n-plane, with equality iff TT special Lagrangian. Hence

$[\operatorname{Re} \Omega] \cdot [L] = \int_L \operatorname{Re} \Omega \leq \int_L \operatorname{vol}_g = \operatorname{vol}(L)$ with equality iff $S$-Lag.
For a general $L \subset X$, $e^{i\theta} : L \to S^1$ may not lift to $\psi : L \to \mathbb{R}$. Obstruction = homotopy class in $[L, S^1] = H^1(L, \mathbb{Z})$.
Up to factor of 2 this is exactly the Narain class $\mu_L$.
For $L$ special Lagrangian, $\mu_L = 0$ automatically ($\Rightarrow$ graded lifts exist
(CF are $Z$-graded).

Deformation of special Lagrangians:

$$ L_t = \exp(t\nu), \nu \in \text{Conf}(NL) \text{ mod value field} $$

$Q^\nu$: when is $L_t$ special Lagrangian? $\nu_t = \exp(t\nu) : L \to X$

$L_t = \nu_t(L)$. $\omega = \nu\omega$

1st order condition: $\frac{d}{dt} \left( \nu^e_\omega \right)_{\mid t=0} = L\nu \omega = d\omega$.

$\beta = -i\nu \omega \in \Omega^1(L, \mathbb{R})$ should be closed $d\beta = 0$

Special: need $\text{Im} \nu_{\mid L} = 0$ ie $\nu^e_\omega (\text{Im} \nu) = 0$

1st order: $\frac{d}{dt} \left( \nu^e \text{Im} \nu \right) = L\nu \text{Im} \nu = d(\nu \text{Im} \nu)$

$\beta_t = \nu \text{Im} \nu \in \Omega^{n-1}(L, \mathbb{R})$ should also be closed $d\beta_t = 0$

Relation between $\beta, \beta_t$? go back to primitive linear algebra:

$$ T_p \mathbb{C}^n \subset \mathbb{C}^n, J_0, \omega_0, T_p L = \mathbb{R}^n, \Omega_p = \psi dx_1 \wedge \ldots dx_n$$

$$ v = \Sigma a_i \frac{\partial}{\partial y_i} \Rightarrow \beta = \Sigma a_i dx_i$$

$$ \beta_t = \Sigma a_i \cdot (-1)^{i-1} \psi dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots dx_n$$

Hence $\beta_t = \psi * \beta$. (Hodge $*$ for $g_{\mid L}$)

In short CY case, $\beta_t = * \beta$, so $d\beta = d\beta_t = 0 \Leftrightarrow \beta$ harmonic.
Prop: 1st order deformations of a special Lagrangian submanifold $\sum (L, R)$ in a strict CY

In a strict CY, 1st order deformations $\mathcal{H}_\psi^1(L, R) = \{ \beta \in \mathcal{O}^1(L, R) \mid d\psi|_\beta = 0, \theta^\ast(\psi|_\beta) = 0 \}$

still true that every class in $\mathcal{H}^1(L, R) \Rightarrow$ unique $\psi$-harm. representation.

(Idea: redo Hodge decomposition but with $\bigwedge \delta^1 \frac{d\psi}{d\psi^1} \rightarrow \bigwedge \delta^2 \frac{d\psi}{d\psi^2}$

$= (d, d^\ast) + \text{order } 0$

or... if dim $n \neq 2$, $\psi$-harmance for $g \Leftrightarrow$ harmance for $\frac{2}{n-2} g$)

Thm: (McLean/ Joyce)

Deformations are unobstructed, i.e. moduli space of slags is a smooth manifold $B$ with $T^\perp B \cong \mathcal{H}_\psi^1(L, R)$. (=$\mathcal{H}^1(L, R)$).

Prf: locally near $L$, $\mathrm{deforms} \xrightarrow{\exp}$ normal vector fields. Consider the Banach bundle $E$ over $U = W^{k, p}(L, NL)$ with fiber at $v$ $W^{k-1, p}(L, \Lambda^2 T^\ast L) \oplus W^{k-1, p}(L, \Lambda^{n-2} T^\ast L)$, and the section $s(v) = (\exp(v) \ast \omega, \exp(v) \ast \text{Im } \Omega)$; then $B = s^{-1}(0)$.

$\omega$, Im $\Omega$ closed $\Rightarrow s(v)$ always takes values in closed forms, and looking at Lie derivatives, since $s(0) = 0$, exact forms.

If $E$ Banach subbundle of exact forms, then $s$ is a Fredholm section of $E$, and $ds(0) \circ (\exp)_\ast \beta \mapsto (-d\beta, d(\psi \ast \beta))$ is onto

$T_v^\perp (NL \cong T^\ast L, \nu \mapsto -i \nu(\psi)) \Rightarrow s^{-1}(0)$ smooth.