Lecture 2  —  Deformations of complex structures, Hodge theory

Reference: Gross-Justice, "CY models & related geometries", ch. 14

1. \((X, J)\) almost complex \((J^2 = -1)\) \(\Rightarrow\) \(TX \otimes \mathbb{C} = T^{1,0} X \oplus T^{0,1} X\)

\[ v^{1,0} = \frac{1}{2} (v - i Jv), \quad v^{0,1} = \frac{1}{2} (v + i Jv) \]

Similarly, \(\text{span}(dz_i) \oplus \text{span}(d\bar{z}_i)\)

\[ \wedge^k T^* X = \bigoplus_{p+q = k} \Lambda^p T^* X \quad \text{holomorphic} \]

\((TX, J) = T^{1,0} X\) complex vector bundle

2. Integrability of complex structure \(\Leftrightarrow [T^{1,0}, T^{1,0}] \subset T^{1,0}\)

\(\Leftrightarrow \exists \exists \theta + \bar{\theta} \text{ maps } \Lambda^p \mathbb{C} \to \Lambda^{p+1} \mathbb{C} \oplus \Lambda^{p+1} \mathbb{C}\)

\(\Leftrightarrow \nabla^2 = 0\)

Then \(TX\) and \(\text{holomorphic vector bundles}\)

Dolbeault cohomology: \(E\) holomorphic vector bundle

\[ C^0(X, E) \xrightarrow{\bar{\partial}} \Lambda^0 \Omega^{0,1}(X, E) \xrightarrow{\bar{\partial}} \Omega^{0,2}(X, E) \xrightarrow{\cdots} \Omega^{2,0}(X, E) \xrightarrow{\bar{\partial}} \cdots \]

\(\Rightarrow H^q_{\bar{\partial}}(X, \mathbb{C}) = \ker \frac{\bar{\partial}}{\bar{\partial}} = 0\)

3. Deforming \(J\) to a nearby \(J'\):

\[ \Omega^0_1, J' \subset T^* X \otimes \mathbb{C} = \Omega^0_1 \otimes \Omega^0_1 \] is the graph of a linear map \((-\bar{\partial})\) from \(\Omega^0_1, J' \to \Omega^0_1, J'\). Conversely, recover \(J\) from \(J'\) if \(s\) is small enough.

Then \(\Omega^0_1, J := \text{graph}(-\bar{\partial}), \Omega^0_1, J = \Omega^0_1, J' = \text{graph}(\bar{\partial}), \)

such that \(T^* X \otimes \mathbb{C} = \Omega^0_1, J' \oplus \Omega^0_1, J'\) and \(s, J' = (i, -i)\)

Can also view \(s\) as a section of \(\left(\Omega^0_1, J\right)^* \otimes \Omega^0_1, J = T^*_J \otimes \Omega^0_1, J\)

(1,0)-forms in \((X, J)\) form with \(s\) values in \(T^0 X\)

\[ z_1, \ldots, z_n \text{ local holomorphic coordinates for } (X, J) \Rightarrow s = \sum_{i,j} \frac{\partial s}{\partial z_j} \otimes d\bar{z}_j \]

Then basis of \((1,0)-\text{forms for } J': dz_i - s(dz_i) = dz_i - \sum_{j} s_{i,j} \frac{\partial}{\partial z_j} \]

(0,1)-vector fields \(\frac{\partial}{\partial \bar{z}_k} + s\left(\frac{\partial}{\partial \bar{z}_k}\right) = \frac{\partial}{\partial \bar{z}_k} + \sum_{l} s_{l,k} \frac{\partial}{\partial z_l}\)
Integrability?

\((\bigoplus_{q} \mathcal{O}_{X}^{0,q} \otimes T^{1,0}_{X}, \mathcal{T})\) Delzant complex for \(T^{1,0}_{X}\) on \((X,J)\)

\((\mathcal{T} \not\equiv 0 \text{ on } X \text{ only})\)

can be a Lie bracket \([\alpha \otimes \nu, \alpha' \otimes \nu'] = (\alpha \otimes \nu) \bullet [\nu, \nu']\)

\(\rightarrow\) diff graded Lie algebra (dgLa).

**Prop.** \(J'\) is integrable \(\Leftrightarrow\) \(\bar{\alpha} s + \frac{1}{2} [s, s] = 0\)

**Pf.** want: \(\left[ \frac{\partial}{\partial z_{i}} + \sum_{l} \frac{s_{li}}{l} \frac{\partial}{\partial z_{l}}, \frac{\partial}{\partial z_{j}} + \sum_{l} \frac{s_{lj}}{l} \frac{\partial}{\partial z_{l}} \right] \in T^{0,1}_{X, J'}?\)

\[= \sum_{l} \left( \frac{\partial s_{lj}}{\partial z_{l}} - \frac{\partial s_{lj}}{\partial z_{l}} \right) + \sum_{k,l} \left( s_{ki} \frac{\partial s_{lj}}{\partial z_{k}} - s_{kj} \frac{\partial s_{li}}{\partial z_{k}} \right) \in \text{span} \left( \frac{\partial}{\partial z_{l}} \right) \ldots\]

\(\Rightarrow\) should be zero: want: \(\forall i,j,l,\)

\[\frac{\partial s_{lj}}{\partial z_{l}} - \frac{\partial s_{lj}}{\partial z_{l}} + \sum_{k} \left( s_{ki} \frac{\partial s_{lj}}{\partial z_{k}} - s_{kj} \frac{\partial s_{li}}{\partial z_{k}} \right) = 0\]

\(\text{Coeff. of } \left( \frac{\partial z_{i}}{\partial z_{j}} \right) \otimes \frac{\partial}{\partial z_{l}}\)

\(\frac{1}{2} \text{Coeff. of } \frac{\partial}{\partial z_{i}} \otimes \frac{\partial}{\partial z_{j}} \otimes \frac{\partial}{\partial z_{l}}\)

in \(\bar{\alpha} s\)

\(\text{in } [s, s]\)

* We'd like to understand \(M_{cx}(X) = \{ J \text{ integrable on } X \}/\text{Diff}(X)\) or take its germ near \(X\)

**Cor.** assuming \(Aut(X, J)\) is discrete, near \(J \in \text{universal family}\)

\(\pi: U \subset M_{cx}, \pi, U \text{ complex manifolds, } \pi \text{ holomorphic, fibers of } \pi \text{ are } \sim X\)

any other family near \(J\) is induced by a classifying map \(\pi\) pullback from \(X\).
\begin{itemize}
    \item \{inherently J's\} \overset{\text{locally}}{=} \{ s \in \Omega^0(X, TX^{1,0}) / \overline{\delta s + \frac{1}{2} [s, s]} = 0 \}

    but need to quotient by \textbf{Diff}(X): \quad J \sim \phi^* J

    If \( \phi \) is close to \text{Id}, can be written in local coords. as
    \[
    \phi: (z_1, \ldots, z_n) \mapsto (z_1 + f_1(z, \overline{z}), \ldots, z_n + f_n(z, \overline{z}))
    \]
    then \( \phi^* dz_i = dz_i + \sum_j \left( \frac{\partial f_i}{\partial z_j} dz_j + \frac{\partial f_i}{\partial \overline{z}_j} d\overline{z}_j \right) \) \quad write \( s \) for \( \phi^* J \)

    or:
    \[
    \exists \phi: TX^{1,0} \to \phi^* TX^{1,0}
    \]
    parts of \( dp \) that commute/anticommute w/ \( J \)
    \[
    \exists \phi: TX^{0,1} \to \phi^* TX^{1,0}
    \]
    \[
    \phi^* dz_i = \frac{d z_i \circ \delta \phi}{(1,0)} + \frac{d z_i \circ \overline{\delta \phi}}{(0,1)} = \left( \frac{d z_i \circ \delta \phi}{(1,0)} \right) \cdot (\text{Id} + (\delta \phi)^{-1} \overline{\delta \phi})
    \]
    i.e. \( s = - (\delta \phi)^{-1} \overline{\delta \phi} \)

    \item \textbf{Tangent space} - infinitesimal deformations ("over Spec } \mathbb{A}[t]/t^2")
    \[
    J(t), \quad J(0) = J \quad \Rightarrow \quad s(t) \in \Omega^0(X, TX^{1,0}), \quad \overline{\delta s(t)} + \frac{1}{2} [s(t), s(t)] = 0
    \]
    \[
    \Rightarrow \quad s_1 = \frac{ds}{dt}|_{t=0} \quad \text{satisfies} \quad \overline{\delta s_1} = 0
    \]

    \textbf{Infinitesimal action of diffeomorphisms}: \[
    (\phi_t), \quad \phi_0 = \text{Id}, \quad \frac{d \phi}{dt}|_{t=0} = \nu \quad \text{vector field} \sim \]

    \[
    \frac{d}{dt}|_{t=0} \left( - (\phi_t)^{-1} \overline{\delta \phi_t} \right) = \frac{d}{dt}|_{t=0} (\overline{\delta \phi_t}) = - \overline{\delta \nu}
    \]

    So:
    \[
    \text{First order deformations} \quad \text{Def}_{1,J}(X, J) = \frac{\ker(\overline{\delta}: \Omega^0(X, TX^{1,0}) \to \Omega^{0,2})}{\text{Im}(\overline{\delta}: \text{C}^\infty(X, TX^{1,0}) \to \Omega^{0,1})} = \text{H}^1(X, TX^{1,0})
    \]

    In particular, given a family \( \phi_t \) of deformations of \((X, J)\) param by \( S \)

    \[
    \text{get a map } T_0 S \to \text{H}^1(X, TX) \text{ by looking at 1st order variations of } J....
    \]
\end{itemize}
Another way to think about this: $(x, J)$ complex n-fold = $(\bigcup U_i)/\phi_{ij}$. 

$U_i$ complex charts, $\phi_{ij}: U_{ij} \to U_i$; holomorphic, $\phi_{ij}^{-1}: U_i \to U_{ij}$; \phi_{ij}\phi_{jk} = \phi_{ik}$.

Then, deforming $(x, J) \leftrightarrow$ deform gluing maps $\phi_{ij}$ among holomorphic maps to 1st order, this is given by $U_i$ vector fields $v_{ij}$ on $U_i \cap U_{ij}$.

& should satisfy $v_{ji} = -v_{ij}$, $v_{ij} + v_{jk} = v_{ik}$ on $U_i \cap U_{ij} \cap U_k$.

$\Rightarrow$ Čech 1-cocycle with values in sheaf of holomorphic vector fields.

Mod out by: $v_i: U_i \cong U_i$; define, change $\phi_{ij} \to \psi_i \phi_{ij} \psi_i^{-1}$.

to 1st order, $v_i$ holomorphic vector fields on $U_i$, affect gluing by $v_{ij} = v_i - v_j$ i.e. Čech coboundary.

$\Rightarrow$ get again $H^1(x, TX)$.

Obstruction: given a first-order deform $s_1$, can we find a

actual deform $s(t) = s_1 t + O(t^2)$ (or a formal deform $\sum s_n t^n$)?

Working order by order to solve $\overline{\partial} s(t) + \frac{i}{2} [s(t), s(t)] = 0$:

$\overline{\partial} s_1 = 0$

$\overline{\partial} s_2 + \frac{i}{2} [s_1, s_1] = 0$

$\overline{\partial} s_3 + [s_1, s_2] = 0$

$\cdots$

$\Rightarrow$ need: $[s_1, s_1] \in Im \overline{\partial} \subseteq \Omega^{0,2}(x, TX^{1,0})$?

know: $[s_1, s_1] \in ker \overline{\partial}$ (since $\overline{\partial} s_1 = 0$).

$\Rightarrow$ primary obstruction: class of $[s_1, s_1]$ in $H^2(x, TX^{1,0})$.

If vanishes then $\exists s_2$ s.t. $\overline{\partial} s_2 + \frac{i}{2} [s_1, s_1] = 0$.

Next obstruction: class of $[s_1, s_2]$ in $H^2(x, TX^{1,0})$.

If vanishes then $\exists s_3$ ... and so on.
If it happen that $H^2(X,TX) = 0$ then deformations are undisturbed, 
 i.e., given $s_1$ lifts to all orders! ($\Rightarrow$ actual deformation).
For Calabi-Yaus, in general $H^2(X,TX) \neq 0$, but remarkably:

Then (Bogomol - Tian - Todorov)

\[ X \text{ compact Calabi-Yau with } H^0(X,TX) = 0 \Rightarrow \text{deformations of } X \text{ are undisturbed, i.e. } \forall \text{ } \mathcal{M}(x) \text{ is locally smooth w/ tangent space } = H^1(X,TX) \]

assuming $\text{Aut}(x,y) = 1$

(for CY mfd , $TX \cong \mathbb{R}^{n-1}$, so $H^0(x,TX) = H^{n-1,0} \cong H^{0,1}$ \leftarrow assume 0
\[ \text{V } \text{ } \Rightarrow \text{V } \mathbb{R} \]

\[ H^1(x,TX) = H^{n-1,1} \leftarrow \text{deformation} \]

\[ H^2(x,TX) = H^{n-1,2} \leftarrow \text{distortion} \]

For Calabi-Yaus, we'll reinterpret Kodaira-Spencer map in terms of $\mathcal{M} \in H^{n,0} = H^n(X,\mathcal{O})$. For this we'll need:

Then (Griffiths transversality)

\[ \forall \alpha \in \mathcal{L}^{p,0}(x,J_x) \Rightarrow \exists \frac{\partial}{\partial t} \alpha \in \mathcal{L}^{p,1} + \mathcal{L}^{p+1,0} + \mathcal{L}^{p+1,1} \]

Pf: $(x,J_x)$ locally given by $s(t) \in \mathcal{L}^{p,0}(x,Tx^{t=0})$, $s(0)=0$

In local coords, $T_{J_x}^{1,0} = \text{span} \{ d\bar{z}_{i}^{(t)} = d\bar{z}_{i} - \sum_{j} s_{ij}(t) \text{d}z_{j} \}$ (seen above)

\[ \alpha_{t} = \sum_{\mid \mu \mid = q, \mid \nu \mid = q} \alpha_{i,j}^{\mu}(t) d\bar{z}_{i}^{(t)} \wedge \cdots \wedge d\bar{z}_{i}^{(t)} \wedge \cdots \wedge d\bar{z}_{j}^{(t)} \wedge \cdots 
\]

Take $\frac{\partial}{\partial t} \mid t=0 \text{ & apply product rule: since } s_{ij}(0)=0, \text{ only terms not } \in \mathcal{L}^{p,0}$

are $\alpha_{i,j}^{(0)} d\bar{z}_{i}^{(0)} \cdots (\sum_{j} \frac{\partial s_{ij}}{\partial t} d\bar{z}_{j}^{(t)} \wedge \cdots \wedge d\bar{z}_{i}^{(t)} \wedge d\bar{z}_{j}^{(t)} \wedge \cdots) \in \mathcal{L}^{p+1,0}$

and similarly (differentiating $d\bar{z}_{i}^{(t)}$) terms in $\mathcal{L}^{p+1,1}$. \( \square \)