Recall: equivalence relation on complexes:

\[ C_i \cong C_{i+1} \cong C_{i+2} \rightarrow \ldots \]

**Def.** A chain map (i.e., \( f : D_i \rightarrow D_{i+1} \rightarrow D_{i+2} \rightarrow \ldots \)) is a quasi-isomorphism if the induced maps on cohomology are isomorphisms.

This is stronger than \( H^\bullet(C_i) = H^\bullet(D_i) \).

**Ex.** \( C[x,y] \rightarrow C[x,y] \) and \( C[x,y] \rightarrow 0 \) not quasi-isomorphic as complexes of \( C[x,y] \)-modules even though same \( H^\bullet \).

**Exs.** \( \{ L^1 \rightarrow O_X \} \) and \( O_D \) are quasi-isomorphic, q.i. \( \iff \) kernel map (similarly with other resolutions of coherent sheaves).

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**Def.** An additive category: 

- \( \text{Hom}(A,B) \) abelian groups
  - Composition is distributive (bilinear)
  - 3 direct sums of objects \( A \oplus B \)
  - 3 zero object \( 0 \) (\( \text{hom}(0,A) = \text{hom}(A,0) = 0 \))

- \text{abelian category} = additive cat. s.t. all morphisms have ker & coker

- Everything defined by universal properties, e.g. kernel of \( f : A \rightarrow B \) is \( K \rightarrow A \) s.t. \( g : C \rightarrow A \) factors (uniquely) through \( K \) iff \( f \circ g = 0 \).

- In actual example, ker / coker are always "initial" ones.

- in an abelian cat. we have notions of exact sequence - cohomology of a complex.

**Def.** An abelian category is the bounded derived cat. \( D^b(A) \):

- Objects = bounded (i.e., finite length) chain complexes in \( A \)
- Morphisms = chain maps up to homotopy, localizing w/ quasi-isomorphisms.

- Homotopy: \( \ldots A_i \xrightarrow{d_i-1} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} \ldots \)

- \( f, g \) are homotopic (\( f \sim g \)) if \( \exists h : A \rightarrow B[-1] \) s.t. \( f-g = d_{B} h + h d_{A} \).

Then look at chain maps /
• Equivalently: bounded complexes form a differential graded category
  morphisms = "premaps of complexes" \( \text{Hom}_k(A, B) = \bigoplus \text{Hom}_A(\text{A}_i, \text{B}_{i+k}) \)
  differential = \( f \in \text{Hom}_k(A, B) \Rightarrow \delta(f) = d_B f + (-1)^{k+1} f d_A \).

  Then chain maps = \( \ker(\delta) : \text{Hom}_0 \to \text{Hom}_1 \)
  nullhomotopic = \( \text{Im}(\delta) : \text{Hom}_1 \to \text{Hom}_0 \)

  \( \Rightarrow \) we want to consider \( \text{H}_0 \text{Hom}(A, B) \).

  Localization w.r.t quasi iso: formal invert quasimos, i.e., add extra morphisms \( s^{-1} \) whenever \( s \) is a quasiiso.
  In other terms, \( \text{Hom}_0^*(A, B) = \{ A \overset{s}{\leftarrow} A' \overset{f}{\rightarrow} B \} / \sim \)
  chain map

  [NB: can skip quot by homotopies, because homotopy equivalences are quasi-isomorphisms, but keeping it makes things more explicit].

• Similarly: \( D^+(A), D^-(A) \) (complexes bounded below, bounded above).

Chain and triangles:

• in category of top spaces (or simplicial complexes \( \mathcal{C}_* \)), \( \mathcal{Z} \) ker \& coker \!
  (unless map is a fibration or an inclusion). However, mapping cone acts as both simultaneously:

  \[ f : X \to Y \rightsquigarrow \text{C}_f := (X \times [0, 1]) \cup Y \bigg/ (x, 0) \sim (x', 0) \]
  \[ (x, 1) \sim f(x) \]

  \[ \begin{array}{c}
  \text{X} \\
  \text{Y}
  \end{array} \]

  There are natural maps \( X \to C_f \) (inclusion) and \( C_f \to \Sigma X \) (collapse \( Y \))

  \[ X \overset{f}{\to} Y \overset{i}{\to} C_f \overset{g}{\to} \Sigma X \to \ldots \]

  with composition nullhomotopic, giving long exact sequence

  \[ H_i(X) \to H_i(Y) \to H_i(C_f) \to H_i(\Sigma X) \to H_{i+1}(Y) \to \ldots \]

  if \( X, Y \) simplicial complexes

  \[ \sim C_f \text{ simplicial complex with i-cells=\{cone on (i-)cells of } X, \text{ e=}(\partial X, 0) \]
  \[ \text{i-cells of } Y \]
3) By analogy: $f: A^* \to B^*$ chain map b/w complexes

$\Rightarrow C_f := A[1] \oplus B, \quad d = \begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$

ie $C_f = A[1] \oplus B$

E.g.: if $A, B$ are single objects, Cone $(f: A \to B)$ is just $\{A \to f \to B\}$

We have natural chain maps $B^* \to C_f^*$ (inclusion of $B$ as subcomplex)

$C_f^* \to A^*[1]$ (quiver complex)

(Con check $A^*[1]$ is quasi-isomorphic to mapping cone of $i: B^* \to C_f^*$)

Thus, in derived category we don't have kernels & cokernels, but we have exact triangles $A^* \to B^* \to C^* \to A^*[1]$ (with corresponding long exact seqs. in homology of complexes)

$H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$

$D^b(A)$ is a triangulated category, namely additive category with a shift functor $T = [1]$ and a set of "distinguished triangles" satisfying various axioms, among which:

- $\forall X \in Ob, \quad \exists$ $\overline{\exists}_X$ is a distinguished triangle

- $\forall f: X \to Y, \exists$ dist. triangle $X \xrightarrow{f} Y \xleftarrow{e}$

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Back to Ext's & derived functors:

1) The det. cat. gives a better way to understand derived functors.

Namely: $F: A \to B$ left exact functor b/w abelian categories

- $R \subset A$ is an adapted class of objects if
  - $R$ is stable under direct sums
  - $C^*$ acyclic complex in $R \Rightarrow F(C^*)$ acyclic
  - $H^i(C) = 0$
  - $\forall A \in A, \exists$ inclusion $0 \to A \to R, R \subset R$.

(ex: injectives)
$K^+(R)$ = homotopy category of complexes bounded below of objects in $R$
Complexes = chain maps up to homotopy

Then: $RF$ = composition $D^+(A) \rightarrow K^+(R) \xrightarrow{F} D^+(B)$

The functor $RF: D^+(A) \rightarrow D^+(B)$ is exact, i.e. exact triangle $\rightarrow$ exact triangles.
Then $R^iF = H^i(RF)$ (what $RF$ does for a single object $A \in A$ is exactly what we do to compute $R^iF(A)$ using a resolution by objects of $R$ & applying $F$, except taking cohomology).

2) Let $A, B \in A$ (e.g. $A[k]$), view them as 1-step complexes in degree 0.
$B[k]$ shift $(B[k])^i = B^{i+k}$; so $B[k]$ concentrated in degree $-k$.

Prop: $\hom_{D^b}(A)(A, B[k]) = \mathbb{E}xt^k_U (A, B)$

We can use this to define product on $\mathbb{E}xt^k_U (A, B) \otimes \mathbb{E}xt^l_U (B, C) \rightarrow \mathbb{E}xt^{k+l}_U (A, C)$

as composition in $D^b(A)$

Example: for $k=1$:

\begin{align*}
0 \rightarrow & 0 \rightarrow A \rightarrow 0 \\
\downarrow & \downarrow \\
0 \rightarrow & B \rightarrow 0 \rightarrow 0
\end{align*}

no chain maps, but we're allowed to invert quasi-isomorphism!!

If we have an extension $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$ (s.e.s. in $A$)
then we get maps of complexes $0 \rightarrow 0 \rightarrow C \rightarrow 0$

\begin{align*}
0 \rightarrow & A \rightarrow B \rightarrow 0 \\
\uparrow & f \\
0 \rightarrow & A \rightarrow 0 \rightarrow 0
\end{align*}

which gives an element of $\hom_{D^b(A)}(C, A[1])$ (e.g. $\mathbb{E}xt^1(A, A)$)
(can do the same with higher $\mathbb{E}xt$'s.)

\begin{itemize}
\item 2 ways to understand the proposition:
\item if $A$ has enough injectives, take an injective resolution of $B$ and replace $B$ by quasi-isom. complex (not bounded, but $D^b \rightarrow D^+$ is full and faithful...) then chain maps $I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$ up to homotopy $= \mathbb{H}^k(\hom(A, I_\bullet))$.
\end{itemize}
check definition of Ext as derived functor:

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow A[1] \]

Then get an exact triangle in \( D^b(A) \):

\[ A \rightarrow B \rightarrow C \rightarrow A[1] \]

\( (\varepsilon = \text{extension map as above}) \)

Axioms of triangulated categories:

\[ \text{Prop.} \quad A \rightarrow B \rightarrow C \rightarrow \text{exact triangle, } E \text{ object } \Rightarrow \text{ long exact sequences} \]

\[ \cdots \rightarrow \text{Hom}(E, A[i]) \rightarrow \text{Hom}(E, B[i]) \rightarrow \text{Hom}(E, C[i]) \rightarrow \text{Hom}(E, A[i + 1]) \rightarrow \cdots \]

\[ \cdots \rightarrow \text{Hom}(A[i + 1], E) \rightarrow \text{Hom}(A[i], E) \rightarrow \text{Hom}(A[i], E) \rightarrow \cdots \]

Applying to our case \((A, B, C, E \text{ 1-step complex})\) we get exactly

the defining property of Ext as derived functor of Hom \( \varepsilon \).

(Idea: e.g., exactness at \( \text{Hom}(E, B) \): (same as other parts)

* check \( \varepsilon \varepsilon = 0 \) for any exact triangle:

\[ \begin{array}{c}
\text{A} \overset{\text{id}}{\rightarrow} \text{A} \overset{\text{id}}{\rightarrow} \text{A}[1] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{C} \overset{\varepsilon}{\rightarrow} \text{B} \overset{\text{id}}{\rightarrow} \text{C} \rightarrow \text{A}[1] \\
\end{array} \]

axiom: \( \exists h \text{ st. square commute} \)

\( h \text{ not be } 0 \Rightarrow \varepsilon \varepsilon = 0 \checkmark \)

* now: assume \( f: A \rightarrow B \text{ st. } \varepsilon \varepsilon f = 0 \).

\[ \begin{array}{c}
\text{E} \overset{\text{id}}{\rightarrow} \text{E} \overset{\text{id}}{\rightarrow} \text{E}[1] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{E} \overset{\text{id}}{\rightarrow} \text{E} \overset{\text{id}}{\rightarrow} \text{E}[1] \\
\end{array} \]

\( \exists g \text{ st. square commute} \)

\[ \begin{array}{c}
\text{A} \overset{\varepsilon}{\rightarrow} \text{B} \overset{\varepsilon}{\rightarrow} \text{C} \rightarrow \text{A}[1] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{A} \overset{\varepsilon}{\rightarrow} \text{B} \overset{\varepsilon}{\rightarrow} \text{C} \rightarrow \text{A}[1] \\
\end{array} \]

\( \Rightarrow f = ug \)

Hence \( \ker \varepsilon \beta = \text{Im} \varepsilon \beta \checkmark \)