Cohomology (continued) $\text{Ext}$ groups: right derived functors of $\text{Hom}$.

NB: internal $\text{Hom}(E, F)$ is a sheaf.

extend $\text{Hom}(E, F)$ = global sections of $\text{Hom}$ = a vector space.

In general, a functor $F: \mathcal{C} \to \mathcal{C}'$ is left exact if

$0 \to A \to B \to C \to 0$ short exact seq. $\Rightarrow 0 \to F(A) \to F(B) \to F(C)$

Right derived functors: $R^i F$ s.t.

$0 \to F(A) \to F(B) \to F(C) \to \cdots \Rightarrow 0 \to R^i F(A) \to R^i F(B) \to R^i F(C) \to \cdots$

To compute $R^i F(A)$, resolve $A$ by injective objects (injective: $\text{Hom}(-, I)$)

($\to F$ become exact): $0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$

then get a complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$

The cohomology of this complex gives $R^i F(A) = \frac{\ker(F(I^i) \to F(I^{i+1}))}{\text{im}(F(I^{i-1}) \to F(I^{i-2}))}$

($R^0 F(A) = F(A)$ by left exactness)

Example: sheaf cohomology = right derived functors of global sections.

* $\text{Hom}(E, -)$ (covariant) and $\text{Hom}(-, F)$ (contravariant) are left exact.
  $\text{Ext}^i = R^i \text{Hom}$. In particular:

$0 \to F_1 \to F_2 \to F_3 \to 0 \Rightarrow 0 \to \text{Hom}(E, F_1) \to \text{Hom}(E, F_2) \to \text{Hom}(E, F_3) \to \cdots$

$0 \to E_1 \to E_2 \to E_3 \to 0 \Rightarrow 0 \to \text{Hom}(E_1, F) \to \text{Hom}(E_2, F) \to \text{Hom}(E_3, F) \to \cdots$

In general, compute by resolving $F$ by injectives (quasi-coh).

* Since $\text{Hom} = \text{H}^0 \text{Hom}$, could try to first understand failure of exactness of $\text{Hom}$, then that of global sections.

Fact: if $E$ is locally free (i.e. vector bundle) then $\text{Hom}(E, -)$ is exact.

Then $\text{Ext}^i(E, F) = \text{H}^i(\text{Hom}(E, F))$.

Otherwise, resolve $E$ by locally free sheaves

$0 \to E_n \to \cdots \to E_0 \to E \to 0$ then we can build a complex of sheaves $\text{Hom}(E_n, F) \to \text{Hom}(E_{n-1}, F) \to \cdots \to \text{Hom}(E_0, F)$

whose hypercohomology computes $\text{Ext}^i(E, F)$.
Example: \( E \) loc. free (vector bundle) 
\( O_p \) skyscraper sheaf at a point

- \( \text{Hom}(E, O_p) = E^* \otimes O_p = \text{skyscraper sheaf with stalk } E^*_p \) at \( p \).

\[
\begin{cases} 
\text{Hom}(E, O_p) = E^*_p \\
\text{Ext}^i(E, O_p) = 0 \quad \forall i > 1
\end{cases}
\]

- \( \text{Hom}(O_p, O_p) = O_p \) but this isn't the whole story...

Resolve \( O_p \) by locally free sheaves, e.g. use Koszul resolution.

This is a local thing near \( p \) \( \Rightarrow \) restricting, can assume \( X \) affine.

Then locally, near \( p \) define a section \( s \) of \( V \subset O_X^n \) (in \( \mathbb{A}^n \))

\[
0 \to \Lambda^0 V \xrightarrow{s} \Lambda^1 V \xrightarrow{s} \cdots \xrightarrow{s} O_X \xrightarrow{s} O_p \to 0
\]

apply \( \text{Hom}(\cdot, O_p) \) gives \( \text{Ext}^i(O_p, O_p) \) is hypercohomology of

\[
O_p \to V \otimes O_p \to \cdots \to \Lambda^{n-1} V \otimes O_p \to \Lambda^n V \otimes O_p
\]

i.e., since complex is bi-infinite, \( \text{Ext}^k(O_p, O_p) = \Lambda^k V \).

- Similarly, \( \text{Ext}^k(O_p, E) = \text{hypercohomology of} \)

\[
E \xrightarrow{s} V \otimes E \xrightarrow{s} \cdots \xrightarrow{s} \Lambda^0 V \otimes E \xrightarrow{s} \Lambda^n V \otimes E
\]

can check this is exact except in last map, cokernel = skyscraper sheaf with stalk \( (\Lambda^n V \otimes E)_p \) at \( p \). (In fact this is its Koszul resolution.)

Hence \( \text{Ext}^k(O_p, E) = \Lambda^k V \otimes E^*_p \simeq E^*_p \), all others zero.

Contrast with Serre duality: \( \text{Ext}^i(E, F) \simeq \text{Ext}^{n-i}(F, k_X \otimes E)^\vee \)

Derived categories: slogan: work with complexes up to homotopy.

- Adjoining a category to include complexes of objects makes it
- algebraically better behaved (e.g. derived category is triangulated)
- less sensitive to initial data (can restrict to nice subset of objects)
  (e.g. on a smooth alg. var., cohom sheaves have a finite
  resolution by vector bundles, so can start with vector
  bundle instead of cohom sheaves...)
  (more important for finite category: allow immersed lagrangians?...)

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even if we know how to define general objects, it's usually easier to
replace them by complexes of better-behaved objects.

E.g. $O_D \langle D = s' \rangle \leftrightarrow$ resolve by complex $L^{-1} \to O_X$

or Koszul resolution used above to compute Ext's for $O_D$

Another example: intersection theory works well with complexes of nice objects

$D_1, D_2 \subset X$ smooth cs. surface defined by sections $S_1, S_2$ of $L_1, L_2$

$\Rightarrow$ intersection theory: $[D_1] \cdot [D_2] = c_1(L_1) \cup c_1(L_2), [X] = c_1(L_1|D_2) \cdot [D_2]$

If $D_1 \cap D_2$ then $O_{D_1} \otimes O_{D_2} = O_{D_1 \cap D_2}$ contains "the right information"

can also resolve by complex $L^{-1}_{1D2} \to O_{D_2}$ (call it $O_{D_1 \cap D_2}$)

(= apply $- \otimes O_{D_2}$ to $L^{-1}_i \to O_X$)

But in non-transverse case, e.g. $D_1 = D_2 = D$, $O_D \otimes O_D = O_D$ looks different?

Point: should instead work at level of complexes and apply $- \otimes O_D$

to the resolution $L^{-1} \to O_X$ of $O_D$, to get $L^{-1}_{ID} \to O_D$

Cokernel of $L^{-1}_{ID} \to O_D$ is still $O_D$, but now there's also a kernel,

which is the information we lost...

$$\begin{cases}
\text{information we lost because } - \otimes O_D \text{ is only right exact, so}
0 \to L^{-1} \to O_D \to O_D \to 0 \text{ only yields } L^{-1}_{ID} \to O_D \to O_D \to 0. \text{ By contrast } - \otimes O_D
\end{cases}$$

* However: a same object may have many different resolutions...

when do we want to treat 2 complexes as isomorphic?
Looking at resolutions, it's tempting to think $H^*(\text{complex})$ is what we want,

but this is much too coarse -- less important information.

E.g. Whitehead: $X, Y$ simplicial complexes, simply connected:

then $X \sim Y$ iff. $\exists$ simplicial complex $Z$ & maps $X \to Z \to Y$

h.e.

1) chain maps $C^*(Z)$ are isos on cohomology.

(though $H^*(X) \cong H^*(Y)$ doesn't imply much, e.g. Massey products...)

(plus though $\sim$ need to subdivide $X/Y$ so homotopy equivalence

then can be approximated by a simplicial map)
Def: $C_i \rightarrow D_i$ chain map (i.e. $C_i \rightarrow C_{i+1} \rightarrow C_{i+2} \rightarrow \ldots$)

is a quasi-isomorphism if the induced maps on cohomology are isomorphisms.

This is stronger than $H^*(C_i) \cong H^*(D_i)$.

Ex: $C[[x,y]] \rightarrow C[[x,y]]$ and $C[[x,y]] \rightarrow C$

not quasi-isomorphic as complexes of $C[[x,y]]$-modules even though same $H^*$

Exs: $\{ L^{-1} \rightarrow O_X \}$ and $O_D$ are quasi-isomorphic, quasi=coherent map

(similarly with other resolutions of coherent sheaves).