Recall: Floer complexes $CF^*(L,L') = \bigwedge^n L_n'$ carry differential $m_1$, product $m_2$, & higher operations $\cdots \otimes CF^*(L_{k-1},L_k) \xrightarrow{m_k} CF^*(L_0,L_k)[2-k]$

Look at $J$-hol. maps $D^2$ with $(k+1)$ boundary marked pts

exp. dim $M(P_1\ldots P_k, q, [u], J) = \deg q - (\deg P_1 + \ldots + \deg P_k) + k - 2$

Assuming transversality, $m_k(P_k\ldots P_1) := \sum_{q \in L_0 \cap L_k} \left( \# M(P_1\ldots P_k, q, [u], J) \right) T^3(u)^q$

Get $A_\infty$-relations when consider $\ell$ of 1-dim! families of discs.

**Prop:** Assuming no bubbling of discs/spheres, we have $\forall m \geq 1$, $\forall P_i \in L_i \cap L_i'$,

$$\sum_{k,l \geq 1} (-1)^* \mathcal{M}_l (P_m, \ldots, P_{j+k-1}, m_k(P_{j+k}, \ldots, P_l), P_i, \ldots, P_1) = 0$$

where $* = \deg (P_1) + \ldots + \deg (P_j) + j$

$\implies m_1$ is a differential
$\to m_2$ satisfies Leibniz rule wrt $m_1$

$m_2$ is associative up to homotopy given by $m_3$

**Def.:** $A_{\infty}$-category = linear "category" where morphism spaces are equipped with such algebraic operations $(m_k)_{k \geq 1}$

Fukaya category = $A_{\infty}$-cat. with objects = lagrangians
morphisms = Floer complexes
alg operations = as above.

(many versions with different details - we'll see later).

So far we have at best an $A_{\infty}$-precategory, i.e. morphisms and compositions are defined only for transverse objects.

($CF(L,L) = ???$)
At the homology level, the Donaldson-Fukaya category \((\text{hom} = HF)\) is easier to work with but contains less information in general.

- "Convergent power series" Floer homology:

We've recorded hom. classes with weights \(T^\omega(u)\).

Gromov compactness = \(\Sigma\) may be infinite but well-defined in Novikov ring \(\Lambda\).

Physicists would actually write \(e^{-2\pi i T^\omega(u)}\) and hope for convergence.

Working over \(\Lambda\), from a physicist's perspective, amounts to considering a family of symplectic forms \((M, \omega_t = \omega_u)\) \(\Leftrightarrow T = e^{-2\pi i t}\) near the large volume limit (t-\(\infty\)) and computing fiber homologies for all \(u_t\) simultaneously (for \(t_0\), if radius of convergence is nonzero; or purely as a formal family near large vol. limit).

Beware: even when it is defined, convergent power series \(HF^*\) need not be a Hamiltonian isopericy invariant.

- Twisted coefficients:

   \(L_i\) Lagrangians are equipped with \((E_i, D_i)\to L_i\) vector bundles w/ flat connections (hint of: C vector bundle w/ flat unitary conn., but could generalize to \(\mathbb{C}\)).

   Define \(\text{CF}((L_0, E_0, D_0), (L_i, E_i, D_i)) = \bigoplus_{p \in L_0 \cap L_i} \text{Hom}(E_0, p) \otimes \Lambda\).

   Then given \(P_1, \ldots, P_k\) (\(p_i \in L_{i-1} \cap L_i\)) and \(W_i \in \text{Hom}(E_{i-1} p_i, E_i p_i)\),

   \[m_k(\ldots, W_1):= \sum_{q \in L_0 \cap L_k} (\# M_{q} P_1 \ldots P_k q, [u], J) \cdot T^\omega(u_j) \cdot P_1 \ldots P_k q \in \text{Hom}(E_0 q, E_k q)\]

   For parallel transport along \(\gamma\) from \(q\) to \(p_1\) using \(D_0\) gives \(\gamma_0 \in \text{Hom}(E_0 q, E_0 p_1)\).

   \(P_i \to P_{i+1}\) \(D_i\) \(g_i \in \text{Hom}(E_i p_i, E_{i+1} p_{i+1})\)

   \(P_k \to q\) \(D_k\)
Flatness of $D_i$ depends only on homotopy class of $u$.

$\nabla_l \cdot w_1 \cdot w_2 \cdot \ldots \cdot w_k \cdot w_0 \in \text{Hom}(E_0, q) (E_k, q)$

Esp. implies to $w_0$: $E_i \rightarrow \text{trivial line bundle } \mathbb{C} \times L_i$

$D_i = \text{Flat } U(1) \text{ connection } D_i = d + iA_i$, $A_i$ closed 1-form

Then: $CF = \bigoplus_{p \in L_0 \times L_1} \bigwedge_{p \in p}$

- (generator $p$, $w = \text{Id}$: $E_0 \cong E_1$)

$\Rightarrow \text{mk groups disc with weights } -\omega(u) \text{ hol}(\omega u)$

Where $\text{hol}(\omega u) \in U(1)$ = holonomy for parallel transport around loop $\omega u$, defined using identification at corners $= \exp(i \sum_{j=0}^{k} (\omega u) A_j)$

- First iteration of Fukaya category (as an $A_\infty$-precat.)

- Objects $= \mathcal{X} = (L, E, V)$,

  - $L$ compact spin Lagrangian ($\mathbb{Z}$-graded version: $\mu_L = 0$, + grading data)
  
  - s.t. $L$ doesn't bound holom. discs.

- $(E, V)$ flat hermitian vector bundle

- for $L_0 \times L_1$, $\text{hom}(L_0, L_1) = CF^{\ast}$ Floer complex

- for transverse sequence, $u_k = \text{operations on Floer complex}$

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2 major outstanding issues:

- $L_0$ not transverse to $L_1$? $\ast$ $L$ bounds disc?

1) what to do if $L_0, L_1$ not transverse? in particular, $CF^{\ast}(L_0, L_1)$?

Various approaches in literature:

a) pick a Hamiltonian perturbation to make them transverse.

(i.e., define $CF^{\ast}(L_0, L_1)$ to be generated by $L_0 \cap \Psi_t(L_1)$ where $H = \Psi(L_0, L_1)$, and perturb all holom curve equations by suitable Hamiltonian terms — in particular, in strip like ends, want $H \rightarrow H(L_{i-1}, L_i)$

See e.g. Seidel's book.
Main issues: (at chain level! then homology easier...)
- need to fix consistent choices of perturbation data.
  - need a procedure which associates to each pair \((L, L')\)
    a Hamiltonian \(H(L, L')\), and to each sequence \((L_0, ..., L_k)\),
    perturbation data for \((k+1)\)-marked holomorphic discs s.t. converges
    to \(H(L_{i-1}, L_i)\) in each strip-like end)
- show different choices yield equivalent categories
- no canonical strict unit \(1 \in CF^*(L, L)\). (only a homology unit)

b) "Morse-Bott" Floer homology (e.g. FOoo)
- \(CF^*(L, L) := C_*(L; \Lambda)\) "singular chains" on \(L\) (in a suitable sense...)
  - operad\( m_k:\) instead of a strip-like end \(\bigotimes\),
    put a boundary marked point \(z\) and require \(u(z) \in \text{Chain}\).
  - E.g.: product \(m_2\) considers
    \[
    \begin{array}{c}
    z_1 \\
    \downarrow \bigotimes \downarrow \\
    z_0 \quad L \quad z_2 \\
    \end{array}
    \]
  - \(ev_i : \mathcal{M}_\beta^0(x; L; J; \beta) \to L\)
  - \(m_2(C_2, C_1) = \sum_{\beta \in \mathcal{M}_\beta^0(x, L)} ev_0 \left[ \mathcal{M}_\beta^0(x, L; J; \beta) \cap ev_1^*C_1 \cap ev_2^*C_2 \right] T^{\omega(\beta)}\)
  - contribution of contact disc \(=\) intersection, product \(C_*(L)\)
    (exception for \(m_2\): don't count contact \(\bigotimes\), instead \(\exists C\) as a chain)
- more generally, if \(L_0, L_1\) have "clean intersection", i.e. \(L_0 \cap L_1\) smooth
  and \(L_0, L_1\) transverse in normal direction to \(L_0 \cap L_1\), want to set
  \(CF^*(L_0, L_1) = C_*(L_0 \cap L_1; \Lambda)\) & use chain as incidence condition
  at strip-like end — analytical details not completely clear.