\( L_0, L_1 \subset (\mathbb{R}, \omega) \) transverse Lagrangians \( \rightarrow CF(L_0, L_1) = \Lambda^{[L_0, L_1]} \)

with differential
\[
\mathcal{D}(p) = \sum_{q \in L_0 \cap L_1, \Phi \in \pi_2^s(\mathbb{R}) \cap M(p, q, \Phi, \mathcal{J})} (\# M(p, q, \Phi, \mathcal{J})) \cdot \nu(\Phi) \cdot q
\]

where \( M = \{ \text{finite energy} \ J\text{-hol. maps} \ u: \mathbb{R} \times [0, 1] \rightarrow M \}
\)
\[ u(s, 0) \in L_0, \ u(s, 1) \in L_1, \ \lim_{s \rightarrow +\infty} u = p, \ \lim_{s \rightarrow -\infty} u = q \]

Limits of sequences in \( M \) have:

- sphere bubbling \( (\text{dim} M = 2 \text{ if transv}) \)
- disc bubbling \( (\text{dim} M = 1 \text{ if transv}) \)
- broken ships

We've seen: if there is no bubbling \( (\text{e.g. if } \nu : \pi_2(M, L_1) = 0) \) then \( \mathcal{D}^2 = 0 \) (by considering ends of moduli spaces of index 2 ships).

*Bubbling of disc is not just a technical issue to overcome, it's an actual obstruction to defining Floer homology.*

**Example:** \( T^2 S^1 = \)

\[
\text{CF}(L_0, L_1) = \Lambda_p \oplus \Lambda_q
\]

\( \mathcal{D}p = \pm T \cdot \nu(u) \cdot q \)

\( \mathcal{D}q = \pm T \cdot \nu(u) \cdot p \)

so... \( \mathcal{D}^2 \neq 0 \)!

What goes wrong: looks at moduli space of index 2 ships from \( p \) to itself. It's an interval...

(Parametrized by upper half-disc \( \bigcirc \),

& setting \( L_1 = \text{unit circle}, \ L_0 = \text{real axis} \),

there are: \( u_\alpha(z) = \frac{z^2 + \alpha}{1 + \alpha z^2}, \ \alpha \in (-1, 1) \)

The two end points:
1) \( q \underset{p}{\xrightarrow{\text{broken trajectory} \ p \rightarrow q \rightarrow p}} \)
2) \( q \underset{p}{\xrightarrow{\text{contracting at} \ p \ \text{and disc bubble with boundary in} \ L_1}} \)

So the disc bubble prevents \( \mathcal{D}^2 = 0 \)
2. Manifold isotopy invce: say $H : [0,1] \times M \to \mathbb{R}$ generate 
$\phi_t^H = \text{flow of } X_H \ (i_X H \omega = dt)$

Consider finite energy solutions of

$$
\begin{align*}
\begin{cases}
u : [0,1] \to M \\
\frac{d\nu}{ds} + \mathcal{J} \left( \frac{d\nu}{dt} - \beta(s) X_H(t, u) \right) = 0 \\
u(s, 0) \in L_0, \ u(s, 1) \in L_1
\end{cases}
\end{align*}
$$

where $\beta = \text{cutoff function}$

\[\begin{array}{c}
L_1 \\
\Downarrow \\
\beta(s) \\
\Updownarrow \\
\Rightarrow \Rightarrow \\
0 \\
\Rightarrow 0 \Rightarrow s
\end{array}\]

cv to trajectory $\gamma(t) = X_H(t, \gamma_0)$

$\gamma(0) \in L_0, \ \gamma(1) \in L_1$

$\Leftrightarrow \gamma(1) = \nu \in \phi_t^H(L_0) \cap L_1$

(set $\nu(s, t) = \phi_t^H(u(s, t))$ then satisfies unperturbed $\mathcal{J}$-eqs for $s \ll 0$).

Gaining index 0 solutions give $\Psi_t^H : \text{CF}(L_0, L_1) \to \text{CF}(\phi_t^H(L_0), L_1)$

(isolated; no $R$-branch. invce now!)

In absence of disc bubbling, can show this is a chain map $\Psi_t^H \circ \partial = \partial' \circ \Psi_t^H$.

(idea: look at ends of index 1 moduli spaces = if no disc bubbling, they must be broken trajectories)

+ this chain map induce an isomorphism in homology.

(idea: look at $\Psi_t^H$ and $\Psi_{-t}^H$ for reversed isotopy, then build a homotopy between $\Psi_t^H \cdot \Psi_{-t}^H$ and $\text{Id} \ldots$).

* More about grading: want gradings on $\text{CF}(L_0, L_1)$ st. $\deg(p) - \deg(p) = \text{index}$?

Recall Madsen index $\Rightarrow \pi_*(\Lambda Gr) = \mathbb{Z}$. Things are easier if $c_1(M) = 0$ (or trivial on $\pi_2$)

then $\Lambda Gr$-bundle of Lagr. planes over $M$ admits a fiberwise universal cover $\Lambda Gr$-bundle of "graded Lagr. planes". Then, if at $p$ we fix graded lifts of $T_p L_i$ we can define the Madsen index of the intersection at $p$. 
If $L_i$ is slightly clockwise from $L_0$, then set $\text{deg}(p) = 0$.

Otherwise, set $\text{deg}(p) = \text{Maslov index from this reference configuration}$.

Obstruction to defining globally graded lift of $L_i = \text{Maslov class} M_i \in H^*(L, \mathbb{R})$. If it vanishes then $\text{ind}(u) = \text{deg}(p) - \text{deg}(q)$ depends only on $p, q$, not on the homotopy class $[u] \Rightarrow$ Floer homology is $\mathbb{Z}$-graded.

Otherwise HF is only $\mathbb{Z}/2\mathbb{Z}$-graded.

Note: if $L_i$ are oriented, then grading mod $2 \equiv \text{sign of intersection}$.

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**Example:**

$M = T^*N$, $\omega = \Sigma dp_i \wedge dq_i$.

Equip $N$ with a Riemann metric $g$, induces metric $\&$ a.c.i.s. on $T^*N$.

Along zero section, $TM = TN \oplus T^*N$, $T^*N \cong TN$ via $\theta$.

Then $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$

$L_0 =$ zero section, $L_1 =$ graph($\varepsilon$ df), $f =$ Morse function on $N$

$\bullet$ $L_0 \cap L_1 =$ cut $(f)$

Maslov index $\Leftrightarrow n -$ Morse index of cut $p$.

(Fukaya-Oh) For $\varepsilon \to 0$, holom. maps $L_0 \to L_1$ graded flow trajectories

$\Rightarrow \text{HF}(L_0, L_1) \cong H_{M_{n-k}}(f) (\cong H^*(N))$

(can discard non-flat geodesics at all)

(geodesics $p \to q$ have $Su^*\omega = \varepsilon (f(q) - f(p))$)

* by Weinstein nbd then, this is a union local model for $L \subset M$ & a $C^1$-small Hamiltonian deformation of $L$. By Hamilton's inv $\varepsilon$ of HF, set $\text{HF}(L, L) = \text{HF}(L, y(L))$. If $L$ doesn't bound disc we conclude $\text{HF}(L, L) \cong H^*(L)$.
If \( L \) does bound discs, but under a suitable assumption to ensure \( \text{HF} \) well-defined, e.g. \( L \) monotone i.e. \( c_1 \) and Maslov positively proportional on \( \pi_2(M, L) \), we have a filtration of Floer complex & a spectral sequence starting with \( \Omega^*(L; \Lambda) \) and converging to \( \text{HF}(L, L) \) (with successive differ = contribution of holom discs of increasing area). \( \Rightarrow \) Oh spectral sequence.