Vector fields    Lecture 2

Let $U$ be an open subset of $\mathbb{R}^n$ and $v$ a vector field on $U$. We’ll say that $v$ is complete if, for every $p \in U$, there exists an integral curve, $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$, i.e., for every $p$ there exists an integral curve that starts at $p$ and exists for all time. To see what “completeness” involves, we recall that an integral curve

$$\gamma : [0, b) \to U,$$

with $\gamma(0) = p$, is called maximal if it can’t be extended to an interval $[0, b')$, $b' > b$. For such curves we showed that either

i. $b = +\infty$
   or

ii. $|\gamma(t)| \to +\infty$ as $t \to b$
   or

iii. the limit set of

$$\{\gamma(t), \ 0 \leq t, b\}$$

contains points on $\partial U$.

Hence if we can exclude ii. and iii. we’ll have shown that an integral curve with $\gamma(0) = p$ exists for all positive time. A simple criterion for excluding ii. and iii. is the following.

**Lemma 1.** The scenarios ii. and iii. can’t happen if there exists a proper $C^1$-function, $\varphi : U \to \mathbb{R}$ with $L_v \varphi = 0$.

**Proof.** $L_v \varphi = 0$ implies that $\varphi$ is constant on $\gamma(t)$, but if $\varphi(p) = c$ this implies that the curve, $\gamma(t)$, lies on the compact subset, $\varphi^{-1}(c)$, of $U$; hence it can’t “run off to infinity” as in scenario ii. or “run off the boundary” as in scenario iii. \qed

Applying a similar argument to the interval $(-b, 0]$ we conclude:

**Theorem 2.** Suppose there exists a proper $C^1$-function, $\varphi : U \to \mathbb{R}$ with the property $L_v \varphi = 0$. Then $v$ is complete.

**Example.**

Let $U = \mathbb{R}^2$ and let $v$ be the vector field

$$v = x^3 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$
Then \( \varphi(x, y) = 2y^2 + x^4 \) is a proper function with the property above.

If \( v \) is complete then for every \( p \), one has an integral curve, \( \gamma_p : \mathbb{R} \to U \) with \( \gamma_p(0) = p \), so one can define a map

\[
f_t : U \to U
\]

by setting \( f_t(p) = \gamma_p(t) \). If \( v \) is \( C^{k+1} \), this mapping is \( C^k \) by the smooth dependence on initial data theorem, and by definition \( f_0(p) = \gamma_p(0) = p \).

We claim that the \( f_t \)'s also have the property

\[
f_t \circ f_a = f_{t+a}.
\]

Indeed if \( f_a(p) = q \), then by the reparameterization theorem, \( \gamma_q(t) \) and \( \gamma_p(t + a) \) are both integral curves of \( v \), and since \( q = \gamma_q(0) = \gamma_p(a) = f_a(p) \), they have the same initial point, so

\[
\gamma_q(t) = f_t(q) = (f_t \circ f_a)(p) = \gamma_p(t + a) = f_{t+a}(p)
\]

for all \( t \). Since \( f_0 \) is the identity it follows from (1) that \( f_t \circ f_{-t} \) is the identity, i.e.,

\[
f_{-t} = f_t^{-1},
\]

so \( f_t \) is a \( C^k \) diffeomorphism. Hence if \( v \) is complete it generates a “one-parameter group”, \( f_t, -\infty < t < \infty \), of \( C^k \)-diffeomorphisms.

For \( v \) not complete there is an analogous result, but it’s trickier to formulate precisely. Roughly speaking \( v \) generates a one-parameter group of diffeomorphisms, \( f_t \), but these diffeomorphisms are not defined on all of \( U \) nor for all values of \( t \). Moreover, the identity (1) only holds on the open subset of \( U \) where both sides are well-defined.

I’ll devote the second half of this lecture to discussing some properties of vector fields which we will need to extend the notion of “vector field” to manifolds. Let \( U \) and \( W \) be open subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively, and let \( f : U \to W \) be a \( C^{k+1} \) map. If \( v \) is a \( C^k \)-vector field on \( U \) and \( w \) a \( C^k \)-vector field on \( W \) we will say that \( v \) and \( w \) are “\( f \)-related” if, for all \( p \in U \) and \( q = f(p) \)

\[
df_p(v_p) = w_q.
\]

Writing

\[
v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}, \quad v_i \in C^k(U)
\]

and

\[
\text{2}
\]
\[ w = \sum_{j=1}^{m} w_j \frac{\partial}{\partial y_j}, \quad w_j \in C^k(V) \]

this equation reduces, in coordinates, to the equation

\[ w_i(q) = \sum \frac{\partial f_i}{\partial x_j}(p)v_j(p). \quad (3) \]

In particular, if \( m = n \) and \( f \) is a \( C^{k+1} \) diffeomorphism, the formula (3) defines a \( C^k \)-vector field on \( V \), i.e.,

\[ w = \sum_{j=1}^{n} w_i \frac{\partial}{\partial y_j} \]

is the vector field defined by the equation

\[ w_i(y) = \sum_{j=1}^{n} \left( \frac{\partial f_i}{\partial x_j}v_j \right) \circ f^{-1}. \quad (4) \]

Hence we’ve proved

**Theorem 3.** If \( f : U \to V \) is a \( C^{k+1} \) diffeomorphism and \( v \) a \( C^k \)-vector field on \( U \), there exists a unique \( C^k \) vector field, \( w \), on \( W \) having the property that \( v \) and \( w \) are \( f \)-related.

We’ll denote this vector field by \( f_*v \) and call it the *push-forward of \( v \) by \( f \).*

I’ll leave the following assertions as easy exercises.

**Theorem 4.** Let \( U_i, \ i = 1, 2, \) be open subsets of \( \mathbb{R}^n \), \( v_i \) a vector field on \( U_i \) and \( f : U_1 \to U_2 \) a \( C^1 \)-map. If \( v_1 \) and \( v_2 \) are \( f \)-related, every integral curve

\[ \gamma : I \to U_1 \]

of \( v_1 \) gets mapped by \( f \) onto an integral curve, \( f \circ \gamma : I \to U_2, \) of \( v_2 \).

**Corollary 5.** Suppose \( v_1 \) and \( v_2 \) are complete. Let \( (f_t)_i : U_i \to U_i, \ -\infty < t < \infty, \) be the one-parameter group of diffeomorphisms generated by \( v_i \). Then \( f \circ (f_1)_t = (f_2)_t \circ f. \)

**Theorem 6.** Let \( U_i, \ i = 1, 2, 3, \) be open subsets of \( \mathbb{R}^n \), \( v_i \) a vector field on \( U_i \) and \( f_i : U_i \to U_{i+1}, \ i = 1, 2 \) a \( C^1 \)-map. Suppose that, for \( i = 1, 2 \), \( v_i \) and \( v_{i+1} \) are \( f_i \)-related. Then \( v_1 \) and \( v_3 \) are \( f_2 \circ f_1 \)-related.

In particular, if \( f_1 \) and \( f_2 \) are diffeomorphisms and \( v = v_1 \)

\[ (f_2)_*(f_1)_*v = (f_2 \circ f_1)_*v. \]
The results we described above have “dual” analogues for one-forms. Namely, let $U$ and $X$ be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : U \to V$ be a $C^{k+1}$-map. Given a one-form, $\mu$, on $V$ one can define a one-form, $f^*\mu$, on $U$ by the following method. For $p \in U$ let $q = f(p)$. By definition $\mu(q)$ is a linear map

$$\mu(q) : T_q\mathbb{R}^m \to \mathbb{R}$$

and by composing this map with the linear map

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^n$$

we get a linear map

$$\mu_q \circ df_p : T_p\mathbb{R}^n \to \mathbb{R},$$

i.e., an element $\mu_q \circ df_p$ of $T_p^*\mathbb{R}^n$.

**Definition 1.** The one-form $f^*\mu$ is the one-form defined by the map

$$p \in U \to (\mu_q \circ df_p) \in T_p^*\mathbb{R}^n$$

where $q = f(p)$.

Note that if $\varphi : V \to \mathbb{R}$ is a $C^1$-function and $\mu = d\varphi$ then by (5)

$$\mu_q \circ df_p = d\varphi_q \circ df_p = d(\varphi \circ f)_p$$

i.e.,

$$f^*\mu = d\varphi \circ f.$$  

**Problem set**

1. Let $U$ be an open subset of $\mathbb{R}^n$, $V$ an open subset of $\mathbb{R}^n$ and $f : U \to V$ a $C^k$ map. Given a function $\varphi : V \to \mathbb{R}$ we’ll denote the composite function $\varphi \circ f : U \to \mathbb{R}$ by $f^*\varphi$.

(a) With this notation show that (6) can be rewritten

$$f^* d\varphi = df^*\varphi.$$  

(b) Let $\mu$ be the one-form

$$\mu = \sum_{i=1}^m \varphi_i \, dx_i \quad \varphi_i \in C^\infty(V)$$

on $V$. Show that if $f = (f_1, \ldots, f_m)$ then

$$f^*\mu = \sum_{i=1}^m f^*\varphi_i \, df_i.$$
(c) Show that if \( \mu \) is \( C^k \) and \( f \) is \( C^{k+1} \), \( f^*\mu \) is \( C^k \).

2. Let \( v \) be a complete vector field on \( U \) and \( f_t : U \to U \), the one parameter group of diffeomorphisms generated by \( v \). Show that if \( \phi \in C^1(U) \)

\[
L_v \phi = \left( \frac{d}{dt} f_t^* \phi \right)_{t=0}.
\]

3. (a) Let \( U = \mathbb{R}^2 \) and let \( v \) be the vector field, \( x_1 \partial/\partial x_2 - x_2 \partial/\partial x_1 \). Show that the curve

\[
t \in \mathbb{R} \to (r \cos(t + \theta), r \sin(t + \theta))
\]

is the unique integral curve of \( v \) passing through the point, \( (r \cos \theta, r \sin \theta) \), at \( t = 0 \).

(b) Let \( U = \mathbb{R}^n \) and let \( v \) be the constant vector field: \( \sum c_i \partial/\partial x_i \). Show that the curve

\[
t \in \mathbb{R} \to a + t(c_1, \ldots, c_n)
\]

is the unique integral curve of \( v \) passing through \( a \in \mathbb{R}^n \) at \( t = 0 \).

(c) Let \( U = \mathbb{R}^n \) and let \( v \) be the vector field, \( \sum x_i \partial/\partial x_i \). Show that the curve

\[
t \in \mathbb{R} \to e^t(a_1, \ldots, a_n)
\]

is the unique integral curve of \( v \) passing through \( a \) at \( t = 0 \).

4. Let \( U \) be an open subset of \( \mathbb{R}^n \) and \( F : U \times \mathbb{R} \to U \) a \( C^\infty \) mapping. The family of mappings

\[
f_t : U \to U, \quad f_t(x) = F(x, t)
\]

is said to be a one-parameter group of diffeomorphisms of \( U \) if \( f_0 \) is the identity map and \( f_s \circ f_t = f_{s+t} \) for all \( s \) and \( t \). (Note that \( f_{-t} = f_{t}^{-1} \), so each of the \( f_t \)'s is a diffeomorphism.) Show that the following are one-parameter groups of diffeomorphisms:

(a) \( f_t : \mathbb{R} \to \mathbb{R}, \quad f_t(x) = x + t \)

(b) \( f_t : \mathbb{R} \to \mathbb{R}, \quad f_t(x) = e^t x \)

(c) \( f_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad f_t(x, y) = (\cos tx - \sin ty, \sin tx + \cos ty) \)

5. Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) be a linear mapping. Show that the series

\[
\exp tA = I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \cdots
\]

converges and defines a one-parameter group of diffeomorphisms of \( \mathbb{R}^n \).
6. (a) What are the infinitesimal generators of the one-parameter groups in exercise 13?

(b) Show that the infinitesimal generator of the one-parameter group in exercise 14 is the vector field

$$\sum a_{i,j} x_j \frac{\partial}{\partial x_i}$$

where $[a_{i,j}]$ is the defining matrix of $A$. 