Vector fields **Lecture 2**

Let U be an open subset of $\mathbb{R}^n$ and v a vector field on U. We'll say that v is *complete* if, for every $p \in U$, there exists an integral curve, $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$, i.e., for every p there exists an integral curve that starts at p and *exists for all time*. To see what "completeness" involves, we recall that an integral curve

$$\gamma : [0, b) \to U\,,$$

with $\gamma(0) = p$, is called *maximal* if it can't be extended to an interval $[0, b')$, $b' > b$. For such curves we showed that either

i. $b = +\infty$
or

ii. $|\gamma(t)| \to +\infty$ as $t \to b$
or

iii. the limit set of

$$\{\gamma(t)\,, \quad 0 \le t, b\}$$

contains points on BdU.

Hence if we can exclude ii. and iii. we'll have shown that an integral curve with $\gamma(0) = p$ exists for all positive time. A simple criterion for excluding ii. and iii. is the following.

Lemma 1. *The scenarios ii. and iii. can't happen if there exists a proper C^1-function, $\varphi : U \to \mathbb{R}$ with $L_v\varphi = 0$.*

Proof. $L_v\varphi = 0$ implies that φ is constant on $\gamma(t)$, but if $\varphi(p) = c$ this implies that the curve, $\gamma(t)$, lies on the compact subset, $\varphi^{-1}(c)$, of U; hence it can't "run off to infinity" as in scenario ii. or "run off the boundary" as in scenario iii.

□

Applying a similar argument to the interval $(-b, 0]$ we conclude:

Theorem 2. *Suppose there exists a proper C^1-function, $\varphi : U \to \mathbb{R}$ with the property $L_v\varphi = 0$. Then v is complete.*

Example.

Let $U = \mathbb{R}^2$ and let v be the vector field

$$v = x^3\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\,.$$

Then $\varphi(x,y) = 2y^2 + x^4$ is a proper function with the property above.

If v is complete then for every p, one has an integral curve, $\gamma_p : \mathbb{R} \to U$ with $\gamma_p(0) = p$, so one can define a map

$$f_t : U \to U$$

by setting $f_t(p) = \gamma_p(t)$. If v is C^{k+1}, this mapping is C^k by the smooth dependence on initial data theorem, and by definition f_0 is the identity map, i.e., $f_0(p) = \gamma_p(0) = p$. We claim that the f_t's also have the property

$$f_t \circ f_a = f_{t+a}\,. \tag{1}$$

Indeed if $f_a(p) = q$, then by the reparameterization theorem, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of v, and since $q = \gamma_q(0) = \gamma_p(a) = f_a(p)$, they have the same initial point, so

$$\begin{aligned} \gamma_q(t) &= f_t(q) = (f_t \circ f_a)(p) \\ &= \gamma_p(t+a) = f_{t+a}(p) \end{aligned}$$

for all t. Since f_0 is the identity it follows from (1) that $f_t \circ f_{-t}$ is the identity, i.e.,

$$f_{-t} = f_t^{-1}\,,$$

so f_t is a C^k diffeomorphism. Hence if v is complete it generates a "one-parameter group", f_t, $-\infty < t < \infty$, of C^k-diffeomorphisms.

For v not complete there is an analogous result, but it's trickier to formulate precisely. Roughly speaking v generates a one-parameter group of diffeomorphisms, f_t, but these diffeomorphisms are not defined on all of U nor for all values of t. Moreover, the identity (1) only holds on the open subset of U where both sides are well-defined.

I'll devote the second half of this lecture to discussing some properties of vector fields which we will need to extend the notion of "vector field" to manifolds. Let U and W be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : U \to W$ be a C^{k+1} map. If v is a C^k-vector field on U and w a C^k-vector field on W we will say that v and w are "f-related" if, for all $p \in U$ and $q = f(p)$

$$df_p(v_p) = \mathrm{w}_q\,. \tag{2}$$

Writing

$$v = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\,, \quad v_i \in C^k(U)$$

and

$$\mathrm{w} \;=\; \sum_{j=1}^{m} \mathrm{w}_j \frac{\partial}{\partial y_j}\,, \quad \mathrm{w}_j \in C^k(V)$$

this equation reduces, in coordinates, to the equation

$$\mathrm{w}_i(q) = \sum \frac{\partial f_i}{\partial x_j}(p) v_j(p)\,. \tag{3}$$

In particular, if $m = n$ and f is a C^{k+1} diffeomorphism, the formula (3) defines a C^k-vector field on V, i.e.,

$$\mathrm{w} = \sum_{j=1}^{n} \mathrm{w}_i \frac{\partial}{\partial y_j}$$

is the vector field defined by the equation

$$\mathrm{w}_i(y) = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}\,. \tag{4}$$

Hence we've proved

Theorem 3. *If $f : U \to V$ is a C^{k+1} diffeomorphism and v a C^k-vector field on U, there exists a unique C^k vector field, w, on W having the property that v and w are f-related.*

We'll denote this vector field by $f_* v$ and call it the *push-forward of v by f*.

I'll leave the following assertions as easy exercises.

Theorem 4. *Let U_i, $i = 1, 2$, be open subsets of $\mathbb{R}^{n_i}$, v_i a vector field on U_i and $f : U_1 \to U_2$ a C^1-map. If v_1 and v_2 are f-related, every integral curve*

$$\gamma : I \to U_1$$

of v_1 gets mapped by f onto an integral curve, $f \circ \gamma : I \to U_2$, of v_2.

Corollary 5. *Suppose v_1 and v_2 are complete. Let $(f_i)_t : U_i \to U_i$, $-\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v_i. Then $f \circ (f_1)_t = (f_2)_t \circ f$.*

Theorem 6. *Let U_i, $i = 1, 2, 3$, be open subsets of $\mathbb{R}^{n_i}$, v_i a vector field on U_i and $f_i : U_i \to U_{i+1}$, $i = 1, 2$ a C^1-map. Suppose that, for $i = 1, 2$, v_i and v_{i+1} are f_i-related. Then v_1 and v_3 are $f_2 \circ f_1$-related.*

In particular, if f_1 and f_2 are diffeomorphisms and $v = v_1$

$$(f_2)_* (f_1)_* v = (f_2 \circ f_1)_* v\,.$$

The results we described above have "dual" analogues for one-forms. Namely, let U and X be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : U \to V$ be a C^{k+1}-map. Given a one-form, μ, on V onc can define a one-form, $f^*\mu$, on U by the following method. For $p \in U$ let $q = f(p)$. By definition $\mu(q)$ is a linear map

$$\mu(q) : T_q\mathbb{R}^m \to \mathbb{R} \tag{5}$$

and by composing this map with the linear map

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^n$$

we get a linear map

$$\mu_q \circ df_p : T_p\mathbb{R}^n \to \mathbb{R}\,,$$

i.e., an element $\mu_q \circ df_p$ of $T_p^*\mathbb{R}^n$.

Definition 1. *The one-form $f^*\mu$ is the one-form defined by the map*

$$p \in U \to (\mu_q \circ df_p) \in T_p^*\mathbb{R}^n$$

where $q = f(p)$.

Note that if $\varphi : V \to \mathbb{R}$ is a C^1-function and $\mu = d\varphi$ then by (5)

$$\mu_q \circ df_p = d\varphi_q \circ df_p = d(\varphi \circ f)_p$$

i.e.,

$$f^*\mu = d\varphi \circ f\,. \tag{6}$$

Problem set

1. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^n$ and $f : U \to V$ a C^k map. Given a function $\varphi : V \to \mathbb{R}$ we'll denote the composite function $\varphi \circ f : U \to \mathbb{R}$ by $f^*\varphi$.

   (a) With this notation show that (6) can be rewritten

   $$f^*\, d\varphi = df^*\varphi\,. \tag{6'}$$

   (b) Let μ be the one-form

   $$\mu = \sum_{i=1}^m \varphi_i\, dx_i \qquad \varphi_i \in \mathcal{C}^\infty(V)$$

   on V. Show that if $f = (f_1, \ldots, f_m)$ then

   $$f^*\mu = \sum_{i=1}^m f^*\varphi_i\, df_i\,.$$

(c) Show that if μ is C^k and f is C^{k+1}, $f^*\mu$ is C^k.

2. Let v be a complete vector field on U and $f_t : U \to U$, the one parameter group of diffeomorphisms generated by v. Show that if $\varphi \in C^1(U)$

$$L_v\varphi = \left(\frac{d}{dt} f_t^*\varphi\right)_{t=0} .$$

3. (a) Let $U = \mathbb{R}^2$ and let $\mathfrak{v}$ be the vector field, $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$. Show that the curve

$$t \in \mathbb{R} \to (r\cos(t+\theta)\,,\, r\sin(t+\theta))$$

is the unique integral curve of $\mathfrak{v}$ passing through the point, $(r\cos\theta, r\sin\theta)$, at $t = 0$.

(b) Let $U = \mathbb{R}^n$ and let $\mathfrak{v}$ be the constant vector field: $\sum c_i\partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to a + t(c_1, \ldots, c_n)$$

is the unique integral curve of $\mathfrak{v}$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

(c) Let $U = \mathbb{R}^n$ and let $\mathfrak{v}$ be the vector field, $\sum x_i\partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to e^t(a_1, \ldots, a_n)$$

is the unique integral curve of $\mathfrak{v}$ passing through a at $t = 0$.

4. Let U be an open subset of $\mathbb{R}^n$ and $F : U \times \mathbb{R} \to U$ a $\mathcal{C}^\infty$ mapping. The family of mappings

$$f_t : U \to U\,, \quad f_t(x) = F(x,t)$$

is said to be a *one-parameter group of diffeomorphisms* of U if f_0 is the identity map and $f_s \circ f_t = f_{s+t}$ for all s and t. (Note that $f_{-t} = f_t^{-1}$, so each of the f_t's is a diffeomorphism.) Show that the following are one-parameter groups of diffeomorphisms:

(a) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = x + t$

(b) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = e^t x$

(c) $f_t : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad f_t(x,y) = (\cos t\, x - \sin t\, y\,,\, \sin t\, x + \cos t\, y)$

5. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Show that the series

$$\exp tA = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbb{R}^n$.

6. (a) What are the infinitesimal generators of the one-parameter groups in exercise 13?

   (b) Show that the infinitesimal generator of the one-parameter group in exercise 14 is the vector field

$$\sum a_{i,j} x_j \frac{\partial}{\partial x_i}$$

   where $[a_{i,j}]$ is the defining matrix of A.