Vector fields Lecture 1

In this lecture we'll reformulate the theorems about ODEs that we've been discussing in the last few lectures in the language of vector fields.

First a few definitions. Given $p \in \mathbb{R}^n$ we define the tangent space to $\mathbb{R}^n$ at p to be the set of pairs

$$T_p\mathbb{R}^n = \{(p, \mathrm{v})\}\,; \quad \mathrm{v} \in \mathbb{R}^n\,. \tag{1}$$

The identification

$$T_p\mathbb{R}^n \to \mathbb{R}^n\,, \quad (p, \mathrm{v}) \to \mathrm{v} \tag{2}$$

makes $T_p\mathbb{R}^n$ into a vector space. More explicitly, for v, v_1 and $\mathrm{v}_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define the addition and scalar multiplication operations on $T_p\mathbb{R}^n$ by the recipes

$$(p, \mathrm{v}_1) + (p, \mathrm{v}_2) \quad = \quad (p, \mathrm{v}_1 + \mathrm{v}_2)$$

and

$$\lambda(p, \mathrm{v}) \quad = \quad (p, \lambda\mathrm{v})\,.$$

Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a C^1 map. We recall that the derivative

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^m$$

of f at p is the linear map associated with the $m \times n$ matrix

$$\left[\frac{\partial f_i}{\partial x_j}(p)\right]\,.$$

It will be useful to have a "base-pointed" version of this definition as well. Namely, if $q = f(p)$ we will define

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

to be the map

$$df_p(p, \mathrm{v}) = (q, Df(p)\mathrm{v})\,. \tag{3}$$

It's clear from the way we've defined vector space structures on $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^m$ that this map is linear.

Suppose that the image of f is contained in an open set, V, and suppose $g : V \to \mathbb{R}^k$ is a C^1 map. Then the "base-pointed"" version of the chain rule asserts that

$$dg_q \circ df_p = d(f \circ g)_p\,. \tag{4}$$

(This is just an alternative way of writing $Dg(q)Df(p) = D(g \circ f)(p)$.)

The basic objects of 3-dimensional vector calculus are vector fields, a *vector field* being a function which attaches to each point, p, of $\mathbb{R}^3$ a base-pointed arrow, $(p, \vec{\mathrm{v}})$. The n-dimensional generalization of this definition is straight-forward.

Definition 1. *Let U be an open subset of $\mathbb{R}^n$. A vector field on U is a function, v, which assigns to each point, p, of U a vector $v(p)$ in $T_p\mathbb{R}^n$.*

Thus a vector field is a vector-valued function, but its value at p is an element of a vector space, $T_p\mathbb{R}^n$ that itself depends on p.

Some examples.

1. Given a fixed vector, $\mathrm{v} \in \mathbb{R}^n$, the function
$$p \in \mathbb{R}^n \to (p, \mathrm{v}) \tag{5}$$
is a vector field. Vector fields of this type are *constant* vector fields.

2. In particular let $e_i, i = 1, \ldots, n$, be the standard basis vectors of $\mathbb{R}^n$. If $\mathrm{v} = e_i$ we will denote the vector field (5) by $\partial/\partial x_i$. (The reason for this "derivation notation" will be explained below.)

3. Given a vector field on U and a function, $f : U \to \mathbb{R}$ we'll denote by fv the vector field
$$p \in U \to f(p)v(p)\,.$$

4. Given vector fields v_1 and v_2 on U, we'll denote by $v_1 + v_2$ the vector field
$$p \in U \to v_1(p) + v_2(p)\,.$$

5. The vectors, (p, e_i), $i = 1, \ldots, n$, are a basis of $T_p\mathbb{R}^n$, so if v is a vector field on U, $v(p)$ can be written uniquely as a linear combination of these vectors with real numbers, $g_i(p)$, $i = 1, \ldots, n$, as coefficients. In other words, using the notation in example 2 above, v can be written uniquely as a sum
$$v = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \tag{6}$$
where $g_i : U \to \mathbb{R}$ is the function, $p \to g_i(p)$.

We'll say that v is a C^k vector field if the g_i's are in $C^k(U)$.

A basic vector field operation is *Lie differentiation*. If $f \in C^1(U)$ we define $L_v f$ to be the function on U whose value at p is given by
$$Df(p)\mathrm{v} = L_v f(p) \tag{7}$$
where $v(p) = (p, \mathrm{v})\,, \quad \mathrm{v} \in \mathbb{R}^n$. If v is the vector field (6) then
$$L_v f = \sum g_i \frac{\partial}{\partial x_i} f \tag{8}$$

(motivating our "derivation notation" for v).

Exercise.

Check that if $f_i \in C^1(U)$, $i = 1, 2$, then

$$L_v(f_1 f_2) = f_1 L_v f_2 + f_1 L_v f_2 \,.$$

We now turn to the main object of this lecture: formulating the ODE results of Birkhoff–Rota, Chapter 6, in the language of vector fields.

Definition 2. *A C^1 curve $\gamma : (a, b) \to U$ is an integral curve of v if for all $a < t < b$ and $p = \gamma(t)$*

$$\left(p, \frac{d\gamma}{dt}(t)\right) = v(p)$$

i.e.., if v is the vector field (6) and $g : U \to \mathbb{R}^n$ is the function $(g_1, \ldots, g_n)$ the condition n for $\gamma(t)$ to be an integral curve of v is that it satisfy the system of ODEs

$$\frac{d\gamma}{dt}(t) = g(\gamma(t)) \,. \tag{9}$$

Hence the ODE results of BR, Chapter 6, give us the following theorems about integral curves.

Theorem 1 (Existence)**.** *Given a point $p_0 \in U$ and $a \in \mathbb{R}$, there exists an interval $I = (a - T, a + T)$, a neighborhood, U_0, of p_0 in U and for every $p \in U_0$ an integral curve, $\gamma_p : I \to U$ with $\gamma_p(a) = p$.*

Theorem 2 (Uniqueness)**.** *Let $\gamma_i : I_i \to U$, $i = 1, 2$, be integral curves. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2 \to U$ defined by*

$$\gamma(t) = \begin{cases} \gamma_1(t)\,, & t \in I_1 \\ \gamma_2(t)\,, & t \in I_2 \end{cases}$$

is an integral curve.

Theorem 3 (Smooth dependence on initial data)**.** *Let v be a C^{k+1}-vector field, on an open subset, V, of U, $I \subseteq \mathbb{R}$ an open interval, $a \in I$ a point on this interval and $h : V \times I \to U$ a mapping with the properties:*

(i) $h(p, a) = p$.

(ii) For all $p \in V$ the curve

$$\gamma_p : I \to U \qquad \gamma_p(t) = h(p, t)$$

is an integral curve of v. Then the mapping, h, is C^k.

One important feature of the system (9) is that it is an *autonomous* system of ODE's: the function, $g(x)$, is a function of x alone, it doesn't depend on t. One consequence of this is the following:

Theorem 4. *Let $I = (a, b)$ and for $c \in \mathbb{R}$ let $I_c = (a - c, b - c)$. Then if $\gamma : I \to U$ is an integral curve, the reparameterized curve*

$$\gamma_c : I_c \to U\,, \quad \gamma_c(t) = \gamma(t + c) \tag{10}$$

is an integral curve.

Finally we recall that a C^1-function $\varphi : U \to \mathbb{R}$ is an *integral* of the system (9) if for every integral curve $\gamma(t)$, the function $t \to \varphi(\gamma(t))$ is constant. This is true if and only if for all t and $p = \gamma(t)$

$$0 = \frac{d}{dt}\varphi(\gamma(t)) = (D\varphi)_p\left(\frac{d\gamma}{dt}\right) = (D\varphi)_p(\mathrm{v})$$

where $(p, \mathrm{v}) = v(p)$. But by (6) the term on the right is $L_v\varphi(p)$.

Here we conclude

Theorem 5. *$\varphi \in C^1(U)$ is an integral of the system (9) if and only if $L_v\varphi = 0$.*

We'll conclude this section by discussing a class of objects which are in some sense "dual objects" to vector fields. For each $p \in \mathbb{R}^n$ let $T_p^*\mathbb{R}^n$ be the dual vector space to $T_p\mathbb{R}^n$, i.e., the space of all linear mappings, $\ell : T_p\mathbb{R}^n \to \mathbb{R}$.

Definition 3. *Let U be an open subset of $\mathbb{R}^n$. A* one-form *on U is a function, ω, which assigns to each point, p, of U a vector, ω_p, in $(T_p\mathbb{R}^n)^*$.*

Some examples:

1. Let $f : U \to \mathbb{R}$ be a C^1 function. Then for $p \in U$ and $c = f(p)$ one has a linear map

$$df_p : T_p\mathbb{R}^n \to T_c\mathbb{R} \tag{11}$$

and by making the identification,

$$T_c\mathbb{R} = \{c, \mathbb{R}\} = \mathbb{R}$$

df_p can be regarded as a linear map from $T_p\mathbb{R}^n$ to $\mathbb{R}$, i.e., as an element of $(T_p\mathbb{R}^n)^*$. Hence the assignment

$$p \in U \to df_p \in (T_p\mathbb{R}^n)^* \tag{12}$$

defines a one-form on U which we'll denote by df.

2. Given a one-form ω and a function, $\varphi : U \to \mathbb{R}$ we define $\varphi\omega$ to be the one-form, $p \in U \to \varphi(p)\omega_p$.

3. Give two one-forms ω_1 and ω_2 we define $\omega_1 + \omega_2$ to be the one-form, $p \in U \to \omega_1(p) + \omega_2(p)$.

4. The one-forms $dx_1, \ldots, dx_n$ play a particularly important role. By (11)

$$(dx_i)\left(\frac{\partial}{\partial ij}\right)_p = \delta_{ij}$$

i.e., equals 1 if $i = j$ and zero if $i \neq j$. Thus $(dx_1)_p, \ldots, (dx_n)_p$ are the basis of $(T_p^*\mathbb{R}^n)^*$ *dual* to the basis $(\partial/\partial x_i)_p$. Therefore, if ω is *any* one-form on U, ω_p can be written uniquely as a sum

$$\omega_p = \sum f_i(p)(dx_i)_p\,, \quad f_i(p) \in \mathbb{R}$$

and ω can be written uniquely as a sum

$$\omega = \sum f_i\, dx_i$$

where $f_i U \to \mathbb{R}$ is the function, $p \to f_i(p)$.

Exercise.

Check that if $f : U \to \mathbb{R}$ is a C^1 function

$$df = \sum \frac{\partial f}{\partial x_i}\, dx_i\,.$$

Problem set.

1. Let U be an open subset of $\mathbb{R}^n$ and let $\gamma : [a, b] \to U$, $t \to (\gamma_1(t), \ldots, \gamma_n(t))$ be a C^1 curve. Given $\omega = \sum f_i\, dx_i$, define the *line integral* of ω over γ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^n \int_a^b f_i(\gamma(t))\frac{d\gamma_i}{dt}\, dt\,.$$

Show that if $\omega = df$ for some $f \in \mathcal{C}^\infty(U)$

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))\,.$$

In particular conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

2. Let

$$\omega = \frac{x_1\, dx_2 - x_2\, dx_1}{x_1^2 + x_2^2}$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ be the closed curve, $t \to (\cos t, \sin t)$. Compute the line integral, $\int_\gamma \omega$, and show that it's not zero. Conclude that ω can't be "d" of a function, $f \in \mathcal{C}^\infty(\mathbb{R}^2 - \{0\})$.

3. Let f be the function

$$f(x_1, x_2) = \begin{cases} \arctan \frac{x_2}{x_1}\,,\, x_1 > 0 \\ \frac{\pi}{2}\,,\, x_1 = 0\,,\, x_2 > 0 \\ \arctan \frac{x_2}{x_1} + \pi\,,\, x_1 < 0 \end{cases}$$

where, we recall: $-\frac{\pi}{2} < \arctan t < \frac{\pi}{2}$. Show that this function is $\mathcal{C}^\infty$ and that df is the 1-form, ω, in the previous exercise. Why doesn't this contradict what you proved in exercise 9?