In this lecture we’ll reformulate the theorems about ODEs that we’ve been discussing in the last few lectures in the language of vector fields.

First a few definitions. Given $p \in \mathbb{R}^n$ we define the tangent space to $\mathbb{R}^n$ at $p$ to be the set of pairs

$$T_p \mathbb{R}^n = \{(p, v) \mid v \in \mathbb{R}^n\}.$$  \hfill (1)

The identification

$$T_p \mathbb{R}^n \to \mathbb{R}^n, \quad (p, v) \to v$$  \hfill (2)

makes $T_p \mathbb{R}^n$ into a vector space. More explicitly, for $v, v_1$ and $v_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define the addition and scalar multiplication operations on $T_p \mathbb{R}^n$ by the recipes

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2)$$

and

$$\lambda(p, v) = (p, \lambda v).$$

Let $U$ be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ a $C^1$ map. We recall that the derivative

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^m$$

of $f$ at $p$ is the linear map associated with the $m \times n$ matrix

$$\left[ \frac{\partial f_i}{\partial x_j}(p) \right].$$

It will be useful to have a “base-pointed” version of this definition as well. Namely, if $q = f(p)$ we will define

$$df_p : T_p \mathbb{R}^n \to T_q \mathbb{R}^m$$

to be the map

$$df_p(p, v) = (q, Df(p)v).$$  \hfill (3)

It’s clear from the way we’ve defined vector space structures on $T_p \mathbb{R}^n$ and $T_q \mathbb{R}^m$ that this map is linear.

Suppose that the image of $f$ is contained in an open set, $V$, and suppose $g : V \to \mathbb{R}^k$ is a $C^1$ map. Then the “base-pointed” version of the chain rule asserts that

$$dg_q \circ df_p = d(f \circ g)_p.$$  \hfill (4)

(This is just an alternative way of writing $Dg(q)Df(p) = D(g \circ f)(p)$.)

The basic objects of 3-dimensional vector calculus are vector fields, a vector field being a function which attaches to each point, $p$, of $\mathbb{R}^3$ a base-pointed arrow, $(p, \vec{v})$. The $n$-dimensional generalization of this definition is straight-forward.
Definition 1. Let \( U \) be an open subset of \( \mathbb{R}^n \). A vector field on \( U \) is a function, \( v \), which assigns to each point, \( p \), of \( U \) a vector \( v(p) \) in \( T_p \mathbb{R}^n \).

Thus a vector field is a vector-valued function, but its value at \( p \) is an element of a vector space, \( T_p \mathbb{R}^n \) that itself depends on \( p \).

Some examples.

1. Given a fixed vector, \( v \in \mathbb{R}^n \), the function

\[
  p \in \mathbb{R}^n \rightarrow (p, v) \tag{5}
\]

is a vector field. Vector fields of this type are constant vector fields.

2. In particular let \( e_i, i = 1, \ldots, n \), be the standard basis vectors of \( \mathbb{R}^n \). If \( v = e_i \) we will denote the vector field (5) by \( \partial/\partial x_i \). (The reason for this “derivation notation” will be explained below.)

3. Given a vector field on \( U \) and a function, \( f : U \rightarrow \mathbb{R} \) we’ll denote by \( fv \) the vector field

\[
  p \in U \rightarrow f(p)v(p) \cdot \tag{6}
\]

4. Given vector fields \( v_1 \) and \( v_2 \) on \( U \), we’ll denote by \( v_1 + v_2 \) the vector field

\[
  p \in U \rightarrow v_1(p) + v_2(p) \cdot \tag{7}
\]

5. The vectors, \( (p, e_i), i = 1, \ldots, n \), are a basis of \( T_p \mathbb{R}^n \), so if \( v \) is a vector field on \( U \), \( v(p) \) can be written uniquely as a linear combination of these vectors with real numbers, \( g_i(p), i = 1, \ldots, n \), as coefficients. In other words, using the notation in example 2 above, \( v \) can be written uniquely as a sum

\[
  v = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \tag{8}
\]

where \( g_i : U \rightarrow \mathbb{R} \) is the function, \( p \rightarrow g_i(p) \).

We’ll say that \( v \) is a \( C^k \) vector field if the \( g_i \)'s are in \( C^k(U) \).

A basic vector field operation is Lie differentiation. If \( f \in C^1(U) \) we define \( L_v f \) to be the function on \( U \) whose value at \( p \) is given by

\[
  Df(p)v = L_v f(p) \tag{9}
\]

where \( v(p) = (p, v), \quad v \in \mathbb{R}^n \). If \( v \) is the vector field (6) then

\[
  L_v f = \sum g_i \frac{\partial}{\partial x_i} f \tag{10}
\]
(motivating our “derivation notation” for $v$).

**Exercise.**

Check that if $f_i \in C^1(U), i = 1, 2$, then

$$L_v(f_1 f_2) = f_1 L_v f_2 + f_1 L_v f_2.$$ 

We now turn to the main object of this lecture: formulating the ODE results of Birkhoff–Rota, Chapter 6, in the language of vector fields.

**Definition 2.** A $C^1$ curve $\gamma : (a, b) \to U$ is an integral curve of $v$ if for all $a < t < b$ and $p = \gamma(t)$

$$\left( p, \frac{d\gamma}{dt}(t) \right) = v(p)$$

i.e., if $v$ is the vector field (6) and $g : U \to \mathbb{R}^n$ is the function $(g_1, \ldots, g_n)$ the condition $n$ for $\gamma(t)$ to be an integral curve of $v$ is that it satisfy the system of ODEs

$$\frac{d\gamma}{dt}(t) = g(\gamma(t)).$$

Hence the ODE results of BR, Chapter 6, give us the following theorems about integral curves.

**Theorem 1 (Existence).** Given a point $p_0 \in U$ and $a \in \mathbb{R}$, there exists an interval $I = (a - T, a + T)$, a neighborhood, $U_0$, of $p_0$ in $U$ and for every $p \in U_0$ an integral curve, $\gamma_p : I \to U$ with $\gamma_p(a) = p$.

**Theorem 2 (Uniqueness).** Let $\gamma_i : I_i \to U, i = 1, 2$, be integral curves. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2 \to U$ defined by

$$\gamma(t) = \begin{cases} 
\gamma_1(t), & t \in I_1 \\
\gamma_2(t), & t \in I_2 
\end{cases}$$

is an integral curve.

**Theorem 3 (Smooth dependence on initial data).** Let $v$ be a $C^{k+1}$-vector field, on an open subset, $V$, of $U$, $I \subseteq \mathbb{R}$ an open interval, $a \in I$ a point on this interval and $h : V \times I \to U$ a mapping with the properties:

(i) $h(p, a) = p$.

(ii) For all $p \in V$ the curve

$$\gamma_p : I \to U \quad \gamma_p(t) = h(p, t)$$

is an integral curve of $v$. Then the mapping, $h$, is $C^k$. 

3
One important feature of the system (9) is that it is an autonomous system of
ODE’s: the function, \( g(x) \), is a function of \( x \) alone, it doesn’t depend on \( t \). One
consequence of this is the following:

**Theorem 4.** Let \( I = (a, b) \) and for \( c \in \mathbb{R} \) let \( I_c = (a - c, b - c) \). Then if \( \gamma : I \to U \)
is an integral curve, the reparameterized curve

\[
\gamma_c : I_c \to U, \quad \gamma_c(t) = \gamma(t + c)
\]

is an integral curve.

Finally we recall that a \( C^1 \)-function \( \varphi : U \to \mathbb{R} \) is an integral of the system (9) if for every integral curve \( \gamma(t) \), the function \( t \to \varphi(\gamma(t)) \) is constant. This is true if and only if for all \( t \) and \( p = \gamma(t) \)

\[
0 = \frac{d}{dt} \varphi(\gamma(t)) = (D\varphi)_p \left( \frac{d\gamma}{dt} \right) = (D\varphi)_p(v)
\]

where \((p, v) = v(p)\). But by (6) the term on the right is \( L_v\varphi(p) \).

Here we conclude

**Theorem 5.** \( \varphi \in C^1(U) \) is an integral of the system (9) if and only if \( L_v\varphi = 0 \).

We’ll conclude this section by discussing a class of objects which are in some sense
“dual objects” to vector fields. For each \( p \in \mathbb{R}^n \) let \( T^*_p\mathbb{R}^n \) be the dual vector space to
\( T_p\mathbb{R}^n \), i.e., the space of all linear mappings, \( \ell : T_p\mathbb{R}^n \to \mathbb{R} \).

**Definition 3.** Let \( U \) be an open subset of \( \mathbb{R}^n \). A one-form on \( U \) is a function, \( \omega \),
which assigns to each point, \( p \), of \( U \) a vector, \( \omega_p \), in \((T^*_p\mathbb{R}^n)\).

Some examples:

1. Let \( f : U \to \mathbb{R} \) be a \( C^1 \) function. Then for \( p \in U \) and \( c = f(p) \) one has a
linear map

\[
(df)_p : T_p\mathbb{R}^n \to T_c\mathbb{R}
\]

and by making the identification,

\[
T_c\mathbb{R} = \{c, \mathbb{R}\} = \mathbb{R}
\]

\( (df)_p \) can be regarded as a linear map from \( T_p\mathbb{R}^n \) to \( \mathbb{R} \), i.e., as an element of
\((T^*_p\mathbb{R}^n)\). Hence the assignment

\[
p \in U \to (df)_p \in (T^*_p\mathbb{R}^n)
\]

defines a one-form on \( U \) which we’ll denote by \( df \).
2. Given a one-form $\omega$ and a function, $\varphi : U \to \mathbb{R}$ we define $\varphi \omega$ to be the one-form, $p \in U \to \varphi(p)\omega_p$.

3. Give two one-forms $\omega_1$ and $\omega_2$ we define $\omega_1 + \omega_2$ to be the one-form, $p \in U \to \omega_1(p) + \omega_2(p)$.

4. The one-forms $dx_1, \ldots, dx_n$ play a particularly important role. By (11)

$$(dx_i)\left(\frac{\partial}{\partial x_j}\right)_p = \delta_{ij}$$

i.e., equals 1 if $i = j$ and zero if $i \neq j$. Thus $(dx_1)_p, \ldots, (dx_n)_p$ are the basis of $(T^*_p\mathbb{R}^n)^*$ dual to the basis $(\partial/\partial x_i)_p$. Therefore, if $\omega$ is any one-form on $U$, $\omega_p$ can be written uniquely as a sum

$$\omega_p = \sum f_i(p)(dx_i)_p, \quad f_i(p) \in \mathbb{R}$$

and $\omega$ can be written uniquely as a sum

$$\omega = \sum f_i dx_i$$

where $f_i U \to \mathbb{R}$ is the function, $p \to f_i(p)$.

Exercise.

Check that if $f : U \to \mathbb{R}$ is a $C^1$ function

$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

Problem set.

1. Let $U$ be an open subset of $\mathbb{R}^n$ and let $\gamma : [a, b] \to U$, $t \to (\gamma_1(t), \ldots, \gamma_n(t))$ be a $C^1$ curve. Given $\omega = \sum f_i dx_i$, define the line integral of $\omega$ over $\gamma$ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt} dt.$$

Show that if $\omega = df$ for some $f \in C^\infty(U)$

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a)).$$

In particular conclude that if $\gamma$ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.
2. Let 
\[ \omega = \frac{x_1 \, dx_2 - x_2 \, dx_1}{x_1^2 + x_2^2} \]
and let \( \gamma : [0,2\pi] \to \mathbb{R}^2 - \{0\} \) be the closed curve, \( t \to (\cos t, \sin t) \). Compute the line integral, \( \int_{\gamma} \omega \), and show that it’s not zero. Conclude that \( \omega \) can’t be “d” of a function, \( f \in C^\infty(\mathbb{R}^2 - \{0\}) \).

3. Let \( f \) be the function
\[ f(x_1, x_2) = \begin{cases} \arctan \frac{x_2}{x_1}, & x_1 > 0 \\ \frac{\pi}{2}, & x_1 = 0, \ x_2 > 0 \\ \arctan \frac{x_2}{x_1} + \pi, & x_1 < 0 \end{cases} \]
where, we recall: \( -\frac{\pi}{2} < \arctan t < \frac{\pi}{2} \). Show that this function is \( C^\infty \) and that \( df \) is the 1-form, \( \omega \), in the previous exercise. Why doesn’t this contradict what you proved in exercise 9?