The theory of manifolds **Lecture 4**

A vector field on an open subset, U, of $\mathbb{R}^n$ is a function, v, which assigns to each point, $p \in U$, a vector, $v(p) \int T_p\mathbb{R}^n$. This definition makes perfectly good sense for manifolds as well.

Definition 1. *Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold. A vector field on X is a function, v, which assigns to each point, $p \in X$, a vector, $v(p) \in T_pX$.*

By definition, T_pX is a vector subspace of $T_p\mathbb{R}^N$ and since

$$T_p\mathbb{R}^n = \{(p, \mathrm{v}), \quad \mathrm{v} \in \mathbb{R}^N\}$$

$v(p)$ is an $(n+1)$-tuple

$$v(p) = (p, v_1(p), \ldots, v_N(p)).$$

Let

$$v_i : X \to \mathbb{R}$$

be the function, $p \in X \to v_i(p)$.

Definition 2. *We will say that v is a $\mathcal{C}^\infty$ vector field if the v_i's are $\mathcal{C}^\infty$ functions.*

Hence by Munkres, §16, exercise 3 we can find a neighborhood, U, of X in $\mathbb{R}^N$ and functions, $w_i \in \mathcal{C}^\infty(U)$ such that $w_i = v_i$ on X. Let w be the vector field,

$$w = \sum_{i=1}^{N} w_i \frac{\partial}{\partial x_i}, \tag{1}$$

on U. This vector field has the property that for every $p \in X$, $w(p) = v(p)$ and in particular,

$$w(p) \in T_pX. \tag{2}$$

A vector field, w, on U with the property (2) for every $p \in X$ is said to be *tangent to X*, and we can summarize what we've shown above as the assertion:

Theorem 1. *If v is a $\mathcal{C}^\infty$ vector field on X, then there exists a neighborhood, U, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ vector field, w, on U with the properties*

i. w is tangent to X.

ii. $w|X = v$.

Given a vector field, v, on X there are often some particularly nice choices for this extended vector field, w. For instance if X is compact then by choosing a bump function, $\rho \in \mathcal{C}_0^\infty(U)$ with $\rho = 1$ on a neighborhood of X, and replacing w_i by ρw_i one can arrange that w be compactly supported. More generally we will say that the vector field, v, is compactly supported if the set

$$\text{supp } v = \overline{\{p \in X\,,\, v(p) \neq 0\}}$$

is compact. Suppose this is the case. Then by choosing the bump function ρ above to be 1 on a neighborhood of supp v we can again arrange that the extension, w, of v to U be compactly supported.

Another interesting choice of an extension is the following. Fix $p \in X$ and let V be a neighborhood of p in $\mathbb{R}^N$ and

$$f_i(x) = 0\,, \quad i+1, \ldots, k \tag{3}$$

an independent set of defining equations for X. (See lecture 2. Here k has to be $N - n$ and the f_i's have to be in $\mathcal{C}^\infty(V)$.)

Proposition 2. *There exists a vector field, w, on V such that $v(q) = w(q)$ at points, $q \in X \cap V$ and*

$$L_w f_i = 0 \qquad i = 1, \ldots, k\,. \tag{4}$$

Since the f_i's are independent the $k \times N$ matrix

$$\left[\frac{\partial f_i}{\partial x_j}(p)\right]\,, \quad i \leq j \leq k\,, \quad 1 \leq j \leq N$$

is of rank k and hence some $k \times k$ minor of this matrix is of rank k. By permuting the x_i's if necessary we can assume that the $k \times k$ matrix

$$\left[\frac{\partial f_i}{\partial x_j}(q)\right]\,, \quad 1 \leq i \leq k\,, \quad i \leq j \leq k \tag{5}$$

is of rank k at $p = q$, and hence (shrinking V if necessary) is of rank k for all $q \in V$. Now let u be *any* vector field on V extending v and let $g_i = L_u f_i$. Since the matrix (5) is non-singular, the system of equations

$$\sum_{j=1}^{k} \frac{\partial f_i}{\partial x_j}(q) a_j(q) = g_i(q) \tag{6}$$

is solvable for the a_j's in terms of the g_i's, and the solutions depend on $\mathcal{C}^\infty$ on q. Moreover, if $q \in X$

$$g_i(q) = (L_u f_i)(q) = (df_i)_q(u(q)) = (df_i)_q(v(q))$$

and since $f_i = 0$ on X and $v_i(q) \in T_qX$, the right hand side of this equation is zero. Thus the g_i's are zero on X and so, by (6), the a_i's are zero on X. Now let

$$w = u - \sum_{j=1}^{k} a_j \frac{\partial}{\partial x_j} .$$

Since the a_i's are zero on X, $w = u = v$ on X and by definition

$$L_w f_i = L_u f_i - \sum \frac{\partial f_i}{\partial x_j} a_j = g_i - g_i = 0 .$$

Q.E.D.

We will now show how to generalize to manifolds a number of vector field results that we discussed in the Vector Field segment of these notes. To simplify the statement of these results we will from now on assume that X is a closed subset of $\mathbb{R}^N$.

1 Integral curves

Let I be an open interval. A $\mathcal{C}^\infty$ map, $\gamma : I \to X$ is an integral curve of v if, for all $t \in I$ and $p = \gamma(t)$,

$$\left(p \, , \frac{d\gamma}{dt} \right) = v(p) . \tag{7}$$

We will show in a moment that the basic existence and uniqueness theorems for integral curves that we proved in Vector Fields, Lecture 1, are true as well for vector fields on manifolds. First, however, an important observation.

Theorem 3. *Let U be a neighborhood of X in $\mathbb{R}^N$ and u a vector field on U extending v. Then if $\gamma : I \to U$ is an integral curve of u and $\gamma(t_0) = p_0 \in X$ for some $t_0 \in I$, γ is an integral curve of v.*

Proof. We will first of all show that $\gamma(I) \subseteq X$. To verify this we note that the set $\{t \in I \, , \gamma(t) \in X\}$ is a closed subset of X, so to show that it's equal to I, it suffices to show that it's open. Suppose as above that t_0 is in this set and $\gamma(t_0) = p_0$. Let V be a neighborhood of p_0 in $\mathbb{R}^N$ and (3) a system of defining equations for $X \cap V$. By Proposition 2 there exists a vertex field, w, on V extending v and satisfying the equations

$$L_w f_i = 0 \, , \quad i = 1, \ldots, k .$$

However these equations say that the f_i's are *integrals* of the vector field w, so if

$$\tilde{\gamma}(t) \, , \quad t_0 - \epsilon < t < t_0 \, ,$$

is an integral curve of w with $\tilde{\gamma}(t_0) = p_0 \in X$ then equations (3) are defining equations for $X \cap V$, so this tells us that $\tilde{\gamma}(t) \in X$ for $t_0 - \epsilon < t < t_0 + \epsilon$, and, in particular, if $\tilde{\gamma}(t) = p$,

$$\left(p, \frac{d\tilde{\gamma}(t)}{dt}\right) w(p) = v(p) = u(p)\,,$$

so $\tilde{\gamma}$ is an integral curve of u and hence since $\tilde{\gamma}(t_0) = p_0 = \gamma(t_0)$, it has to coincide with γ on the integral $-\epsilon + t_0 < t < \epsilon + t_0$. This shows that the set of points

$$\{t \in I\,, \quad \gamma(t) \in X\}$$

is open and hence is equal to I.

Finally, to conclude the proof of Theorem 3 we note that since $\gamma(I) \subseteq X$, it follows that for *any* point $t \in I$ and $p = \gamma(t)$

$$\left(p, \frac{d\gamma}{dt}\right) = w(p) = v(p)$$

since $v = w$ on X and $p \in X$. Hence γ is an integral curve of v as claimed.

□

From the existence and uniqueness theorems for integral curves of w we obtain similar theorems for v.

Theorem 4 (Existence)**.** *Given a point, $p_0 \in X$ and $a \in \mathbb{R}$ there exists an interval, $I = (a - T, a + T)$, a neighborhood, U_0, of p_0 in X and, for every $p \in U$, an integral curve, $\gamma_p : I \to X$ with $\gamma_p(q) = p$.*

Theorem 5 (Uniqueness)**.** *Let $\gamma_i : I_i \to X$ $i = 1, 2$ be integral curves of v. Suppose that for some $a \in I$, $\gamma_1(a) = \gamma_2(a)$. Then $\gamma_1 = \gamma_2$ on $I_1 \cap I_2$ and the curve, $\gamma : I_1 \cup I_2 \to X$ defined by*

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in I_1 \\ \gamma_2(t), & t \in I_2 \end{cases}$$

is an integral curve of v.

2 One parameter groups of diffeomorphisms

Suppose that v is compactly supported. Then, as we pointed out above, we can find a neighborhood, U, of X and a vector field, w, on U which is also compactly supported and extends v. Since w is compactly supported, it is complete, so for every $p \in U$ there exists an integral curve

$$\gamma_p(t)\,, \quad -\infty < t < \infty \tag{8}$$

with $\gamma_p(0) = 0$, and the maps

$$g_t : U \to U\,, \quad g_t(p) = \gamma_p(t)$$

are a one-parameter group of diffeomorphisms. In particular if $p \in X$ the curve (8) is, as we've just shown, an integral curve of v and hence $g_t(p) = \gamma_p(t) \in X$ i.e.,

$$f_t \stackrel{\text{def}}{=} g_t|X$$

is a $\mathcal{C}^\infty$ map of X into X. Moreover, $g_t^{-1} = g_{-t}$, so $f_t^{-1} = f_{-t}$, so the f_t's are a one-parameter group of diffeomorphisms of X with the defining property: For all $p \in X$ the curve

$$\gamma_p(t) = f_t(p)\,, \quad -\infty < t < \infty$$

is the unique integral curve of v with $\gamma_p(\sigma) = p$. If X itself is compact then v is automatically compactly supported so we've proved

Theorem 6. *If X is compact,* every *vector field, v, is complete and generates a one-parameter group of diffeomorphisms, f_t.*

3 Lie differentiation

Let $\varphi : X \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. As in Vector Fields, Lecture 1, we will define the Lie derivative

$$L_v\varphi \in \mathcal{C}^\infty(X) \tag{9}$$

by the recipe

$$L_v\varphi = d\varphi_p(v(p)) \tag{10}$$

at $p \in X$. (If U is an open neighborhood of X in $\mathbb{R}^N$ and w and ψ extensions of v and φ to U then $L_v\varphi$ is the restriction of $L_w\psi$ to X_j, so this shows that (9) is indeed in $\mathcal{C}^\infty(X)$.) If $L_v\varphi = 0$ then φ is an *integral* of v and is constant along integral curves, $\gamma(t)$, of v as is clear from the identity

$$\frac{d}{dt}\varphi(\gamma(t)) = L_v\varphi(\gamma(t)) = 0\,. \tag{11}$$

The following we'll leave as an exercise. Suppose v is complete and generates a one-parameter group of diffeomorphisms

$$f_t : X \to X\,, \quad -\infty < t < \infty$$

then

$$L_v\varphi = \frac{d}{dt} f_t^*\varphi|_{t=0} \tag{12}$$

where $f_t^*\varphi = \varphi \circ f_t$.

4 f-relatedness

Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Given vector fields, v and w, on X and Y, we'll say they are f-related if, for every $p \in X$ and $q = f(p)$

$$df_p(v(p)) = w(q)\,. \tag{13}$$

For "f-relatedness" on manifolds one has analogues of the theorems we proved earlier for open subsets of Euclidean space. In particular one has

Theorem 7. *If $\gamma : I \to X$ is an integral curve of v, $f \circ \gamma : I \to Y$ is an integral curve of w.*

Theorem 8. *If v and w are generators of one-parameter groups of diffeomorphisms*

$$f_t : X \to Y$$

and

$$g_t : Y \to Y$$

then $g_t \circ f = f \circ f_t$.

Theorem 9. *If $\varphi \in \mathcal{C}^\infty(Y)$ and $f^*\varphi \in \mathcal{C}^\infty(X)$ is the function $\varphi \circ f$,*

$$D_v f^*\varphi = f^* D_w \varphi\,.$$

The proofs of these three results are, more or less verbatim, the same as before, and we'll leave them as exercises.

If X and Y are manifolds of the same dimension and $f : X \to Y$ is a diffeomorphism then, given a vector field, v, on X one can define, as we did before, a unique f-related vector field, w, on Y by the formula

$$w(q) = df_p(v(p)) \tag{14}$$

where $p = f^{-1}(q)$. We'll denote this vector field by f_*v and, as before, call it the *push-forward of* v by f. We'll leave for you to show that if v is a $\mathcal{C}^\infty$ vector field the vector field defined by (14) is as well.

Our final exercise: Let X_i, $i = 1, 2, 3$ be manifolds, v_i a vector field on X_i and $f : X_1 \to X_2$ and $g : X_2 \to X_3$ $\mathcal{C}^\infty$ maps. Prove

Theorem 10. *If v_1 and v_2 are f-related and v_2 and v_3 are g-related, then v_1 and v_3 are $g \circ f$-related.*

Hint: Chain rule.

Problem Set

1. Let $X \subseteq \mathbb{R}^3$ be the paraboloid, $x_3 = x_1^2 + x_2^2$ and let w be the vector field

$$w = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \partial x_3 \frac{\partial}{\partial x_3} .$$

   (a) Show that w is tangent to X and hence defines by restriction a vector field, v, on X.

   (b) What are the integral curves of v?

2. Let S^2 be the unit 2-sphere, $x_1^2 + x_2^2 + x_3^2 = 1$, in $\mathbb{R}^3$ and let w be the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} .$$

   (a) Show that w is tangent to S^2, and hence by restriction defines a vector field, v, on S^2.

   (b) What are the integral curves of v?

3. As in problem 2 let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and let w be the vector field

$$w = \frac{\partial}{\partial x_3} - x_3 \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right)$$

   (a) Show that w is tangent to S^2 and hence by restriction defines a vector field, v, on S^2.

   (b) What do its integral curves look like?

4. Let S^1 be the unit sphere, $x_1^2 + x_2^2 = 1$, in $\mathbb{R}^2$ and let $X = S^1 \times S^1$ in $\mathbb{R}^4$ with defining equations

$$\begin{aligned} f_1 &= x_1^2 + x_2^2 - 1 = 0 \\ f_2 &= x_3^2 + x_4^2 - 1 = 0 . \end{aligned}$$

   (a) Show that the vector field

$$w = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \lambda \left(x_4 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_4} \right) ,$$

   $\lambda \in \mathbb{R}$, is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

(c) Show that $L_w f_i = 0$.

5. For the vector field, v, in problem 4, describe the one-parameter group of diffeomorphisms it generates.

6. Let X and v be as in problem 1 and let $f : \mathbb{R}^2 \to X$ be the map, $f(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$. Show that if u is the vector field,

$$u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2},$$

then $f_* u = v$.

7. Verify (11).

8. Let X be a submanifold of X in $\mathbb{R}^N$ and let v and w be the vector fields on X and U. Denoting by ι the inclusion map of X into U, show that v and w are ι-related if and only if w is tangent to X and its restriction to X is v.

9. Verify that the vector field (14) is $\mathcal{C}^\infty$.

10.* An elementary result in number theory asserts

Theorem. *A number, $\lambda \in \mathbb{R}$, is irrational if and only if the set*

$$\{m + \lambda n\,, \quad m \text{ and } n \text{ intgers}\}$$

is a dense *subset of* $\mathbb{R}$.

Let v be the vector field in problem 4. Using the theorem above prove that if $\lambda/2\pi$ is irrational then for every integral curve, $\gamma(t)$, $-\infty < t < \infty$, of v the set of points on this curve is a dense subset of X.