The theory of manifolds **Lecture 3**

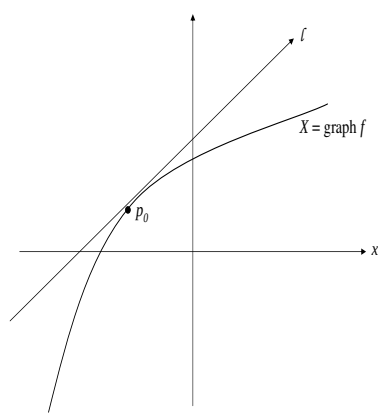
We recall that a subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold, if, for every $p \in X$, there exists an open set, $U \subseteq \mathbb{R}^n$, a neighborhood, V, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$-diffeomorphism, $\varphi : U \to X \cap X$.

Definition 1. *We will call φ a* parameterization *of X at p.*

Our goal in this lecture is to define the notion of the *tangent space*, T_pX, to X at p and describe some of its properties. Before giving our official definition we'll discuss some simple examples.

Example 1.

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $X = \text{graph} f$.



Then in this figure above the tangent line, ℓ, to X at $p_0 = (x_0, y_0)$ is defined by the equation

$$y - y_0 = a(x - x_0)$$

where $a = f'(x_0)$ In other words if p is a point on ℓ then $p = p_0 + \lambda \mathrm{v}_0$ where $\mathrm{v}_0 = (1, a)$ and $\lambda \in \mathbb{R}^n$. We would, however, like the tangent space to X at p_0 to be a subspace of the tangent space to $\mathbb{R}^2$ at p_0, i.e., to be the subspace of the space: $T_{p_0}\mathbb{R}^2 = \{p_0\} \times \mathbb{R}^2$, and this we'll achieve by defining

$$T_{p_0}X = \{(p_0, \lambda \mathrm{v}_0), \quad \lambda \in \mathbb{R}\}.$$

Example 2.

Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. The tangent plane to S^2 at p_0 is usually defined to be the plane

$$\{p_0 + \mathrm{v}\,;\, v \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}\,.$$

However, this tangent plane is easily converted into a subspace of $T_p\mathbb{R}^3$ via the map, $p_0 + \mathrm{v} \to (p_0, \mathrm{v})$ and the image of this map

$$\{(p_0, \mathrm{v})\,;\, \mathrm{v} \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}$$

will be our definition of $T_{p_0}S^2$.

Let's now turn to the general definition. As above let X be an n-dimensional submanifold of $\mathbb{R}^N$, p a point of X, V a neighborhood of p in $\mathbb{R}^N$, U an open set in $\mathbb{R}^n$ and

$$\varphi : (U, q) \to (X \cap V, p)$$

a parameterization of X. We can think of φ as a $\mathcal{C}^\infty$ map

$$\varphi : (U, q) \to (V, p)$$

whose image happens to lie in $X \cap V$ and we proved last time that its derivative at q

$$(d\varphi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N \tag{1}$$

is injective.

Definition 2. *The tangent space, T_pX, to X at p is the image of the linear map (1). In other words, $w \in T_p\mathbb{R}^N$ is in T_pX if and only if $w = d\varphi_q(\mathrm{v})$ for some $\mathrm{v} \in T_q\mathbb{R}^n$. More succinctly,*

$$T_pX = (d\varphi_q)(T_q\mathbb{R}^n)\,. \tag{2}$$

(Since $d\varphi_q$ is injective this space is an n-dimensional vector subspace of $T_p\mathbb{R}^N$.)

One problem with this definition is that it appears to depend on the choice of φ. To get around this problem, we'll give an alternative definition of T_pX. Last time we showed that there exists a neighborhood, V, of p in $\mathbb{R}^N$ (which we can without loss of generality take to be the same as V above) and a $\mathcal{C}^\infty$ map

$$f : (V, p) \to (\mathbb{R}^k, 0)\,, \quad k = N - n\,, \tag{3}$$

such that $X \cap V = f^{-1}(0)$ and such that f is a submersion at all points of $X \cap V$, and in particular at p. Thus

$$df_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$$

is surjective, and hence the kernel of df_p has dimension n. Our alternative definition of T_pX is

$$T_pX = \text{kernel } \, df_p\,. \tag{4}$$

The spaces (2) and (4) are both n-dimensional subspaces of $T_p\mathbb{R}^N$, and we claim that these spaces are the same. (Notice that the definition (4) of T_pX doesn't depend on φ, so if we can show that these spaces are the same, the definitions (2) and (4) will depend *neither* on φ *nor* on f.)

Proof. Since $\varphi(U)$ is contained in $X \cap V$ and $X \cap V$ is contained in $f^{-1}(0)$, $f \circ \varphi = 0$, so by the chain rule

$$df_p \circ df_q = f(f \circ \varphi)_q = 0\,. \tag{5}$$

Hence if $\mathrm{v} \in T_p\mathbb{R}^n$ and $w = d\varphi_q(\mathrm{v})$, $df_p(w) = 0$. This shows that the space (2) is contained in the space (4). However, these two spaces are n-dimensional so the coincide.

□

From the proof above one can extract a slightly stronger result:

Theorem 1. *Let W be an open subset of $\mathbb{R}^\ell$ and $h : (W, q) \to (\mathbb{R}^N, p)$ a $\mathcal{C}^\infty$ map. Suppose $h(W)$ is contained in X. Then the image of the map*

$$dh_q : T_q\mathbb{R}^\ell \to T_p\mathbb{R}^N$$

is contained in T_pX.

Proof. Let f be the map (3). We can assume without loss of generality that $h(W)$ is contained in V, and so, by assumption, $h(W) \subseteq X \cap V$. Therefore, as above, $f \circ h = 0$, and hence $dh_q(T_q\mathbb{R}^\ell)$ is contained in the kernel of df_p.

□

This result will enable us to define the *derivative* of a mapping between manifolds. Explicitly: Let X be a submanifold of $\mathbb{R}^N$, Y a submanifold of $\mathbb{R}^m$ and $g : (X, p) \to (Y, y_0)$ a $\mathcal{C}^\infty$ map. By Theorem 1 there exists a neighborhood, $\mathcal{O}$, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $\tilde{g} : \mathcal{O} \to \mathbb{R}^m$ extending to g. We will define

$$(dg_p) : T_pX \to T_{y_0}Y \tag{6}$$

to be the restriction of the map

$$(d\tilde{g})_p : T_p\mathbb{R}^N \to T_{y_0}\mathbb{R}^m \tag{7}$$

to T_pX. There are two obvious problems with this definition:

1. Is the space
$$(d\tilde{g}_p)(T_pX)$$
contained in $T_{y_0}Y$?

2. Does the definition depend on $\tilde{g}$?

To show that the answer to 1. is yes and the answer to 2. is no, let

$$\varphi : (U, x_0) \to (X \cap V, p)$$

be a parameterization of X, and let $h = \tilde{g} \circ \varphi$. Since $\varphi(U) \subseteq X$, $h(U) \subseteq Y$ and hence by Theorem 2

$$dh_{x_0}(T_{x_0}\mathbb{R}^n) \subseteq T_{y_0}Y \, .$$

But by the chain rule

$$dh_{x_0} = d\tilde{g}_p \circ d\varphi_{x_0} \, , \tag{8}$$

so by (2)

$$(d\tilde{g}_p)(T_pX) \;\subseteq\; T_pY \tag{9}$$

and

$$(d\tilde{g}_p)(T_pX) \;=\; (dh)_{x_0}(T_{x_0}\mathbb{R}^n) \tag{10}$$

Thus the answer to 1. is yes, and since $h = \tilde{g} \circ \varphi = g \circ \varphi$, the answer to 2. is no.

From (5) and (6) one easily deduces

Theorem 2 (Chain rule for mappings between manifolds)**.** *Let Z be a submanifold of $\mathbb{R}^\ell$ and $\psi : (Y, y_0) \to (Z, z_0)$ a $\mathcal{C}^\infty$ map. Then $d\psi_q \circ dg_p = d(\psi \circ g)_p$.*

Problem set

1. What is the tangent space to the quadric, $x_n^2 = x_1^2 + \cdots + x_{n-1}^2$, at the point, $(1, 0, \ldots, 0, 1)$?

2. Show that the tangent space to the $(n-1)$-sphere, S^{n-1}, at p, is the space of vectors, $(p, \mathrm{v}) \in T_p\mathbb{R}^n$ satisfying $p \cdot \mathrm{v} = 0$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let $X = \mathrm{graph} f$. What is the tangent space to X at $(a, f(a))$?

4. Let $\sigma : S^{n-1} \to S^{n-1}$ be the anti-podal map, $\sigma(x) = -x$. What is the derivative of σ at $p \in S^{n-1}$?

5. Let $X_i \subseteq \mathbb{R}^{N_i}$, $i = 1, 2$, be an n_i-dimensional manifold and let $p_i \in X_i$. Define X to be the Cartesian product

   $$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

   and let $p = (p_1, p_2)$. Describe T_pX in terms of $T_{p_1}X_1$ and $T_{p_2}X_2$.

6. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold and $\varphi_i : U_i \to X \cap V_i$, $i = 1, 2$. From these two parameterizations one gets an overlap diagram

$$\begin{tikzcd} & X \cap V & \\ W_1 \arrow[ur, "d_1"] \arrow[rr, "\psi"] & & W_2 \arrow[ul, "d_2"'] \end{tikzcd} \tag{11}$$

where $V = V_1 \cap V_2$, $W_i = \varphi_i^{-1}(X \cap V)$ and $\psi = \varphi_2^{-1} \circ \varphi_1$.

(a) Let $p \in X \cap V$ and let $q_i = \varphi_i^{-1}(p)$. Derive from the overlap diagram (10) an overlap diagram of linear maps

$$\begin{tikzcd} & T_p\mathbb{R}^N \arrow[dr, "(d\varphi_2)_{q_2}"] & \\ T_{q_1}\mathbb{R}^n \arrow[ur, "(d\varphi_1)_{q_1}"] \arrow[rr, "(d\psi)_{q_1}"] & & T_{q_2}\mathbb{R}^n \end{tikzcd} \tag{12}$$

(b) Use overlap diagrams to give another proof that T_pX is intrinsically defined.