The theory of manifolds   Lecture 3

We recall that a subset, $X$, of $\mathbb{R}^N$ is an $n$-dimensional manifold, if, for every $p \in X$, there exists an open set, $U \subseteq \mathbb{R}^n$, a neighborhood, $V$, of $p$ in $\mathbb{R}^N$ and a $C^\infty$-diffeomorphism, $\varphi : U \to X \cap X$.

**Definition 1.** We will call $\varphi$ a parameterization of $X$ at $p$.

Our goal in this lecture is to define the notion of the tangent space, $T_pX$, to $X$ at $p$ and describe some of its properties. Before giving our official definition we’ll discuss some simple examples.

**Example 1.**

Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function and let $X = \text{graph } f$.

Then in this figure above the tangent line, $\ell$, to $X$ at $p_0 = (x_0, y_0)$ is defined by the equation

$$y - y_0 = a(x - x_0)$$

where $a = f'(x_0)$ In other words if $p$ is a point on $\ell$ then $p = p_0 + \lambda v_0$ where $v_0 = (1, a)$ and $\lambda \in \mathbb{R}^n$. We would, however, like the tangent space to $X$ at $p_0$ to be a subspace of the tangent space to $\mathbb{R}^2$ at $p_0$, i.e., to be the subspace of the space: $T_{p_0}\mathbb{R}^2 = \{p_0\} \times \mathbb{R}^2$, and this we’ll achieve by defining

$$T_{p_0}X = \{(p_0, \lambda v_0), \quad \lambda \in \mathbb{R}\}.$$
Example 2.

Let $S^2$ be the unit 2-sphere in $\mathbb{R}^3$. The tangent plane to $S^2$ at $p_0$ is usually defined to be the plane

$$\{p_0 + v; v \in \mathbb{R}^3, \ v \perp p_0\}.$$  

However, this tangent plane is easily converted into a subspace of $T_{p_0}\mathbb{R}^3$ via the map, $p_0 + v \to (p_0, v)$ and the image of this map

$$\{(p_0, v); v \in \mathbb{R}^3, \ v \perp p_0\}$$

will be our definition of $T_{p_0}S^2$.

Let’s now turn to the general definition. As above let $X$ be an $n$-dimensional submanifold of $\mathbb{R}^N$, $p$ a point of $X$, $V$ a neighborhood of $p$ in $\mathbb{R}^N$, $U$ an open set in $\mathbb{R}^n$ and

$$\varphi : (U, q) \to (X \cap V, p)$$

a parameterization of $X$. We can think of $\varphi$ as a $C^\infty$ map

$$\varphi : (U, q) \to (V, p)$$

whose image happens to lie in $X \cap V$ and we proved last time that its derivative at $q$

$$(d\varphi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N$$  \hspace{1cm} (1)

is injective.

**Definition 2.** The tangent space, $T_pX$, to $X$ at $p$ is the image of the linear map (1). In other words, $w \in T_p\mathbb{R}^N$ is in $T_pX$ if and only if $w = d\varphi_q(v)$ for some $v \in T_q\mathbb{R}^n$. More succinctly,

$$T_pX = (d\varphi_q)(T_q\mathbb{R}^n).$$  \hspace{1cm} (2)

(Since $d\varphi_q$ is injective this space is an $n$-dimensional vector subspace of $T_p\mathbb{R}^N$.)

One problem with this definition is that it appears to depend on the choice of $\varphi$. To get around this problem, we’ll give an alternative definition of $T_pX$. Last time we showed that there exists a neighborhood, $V$, of $p$ in $\mathbb{R}^N$ (which we can without loss of generality take to be the same as $V$ above) and a $C^\infty$ map

$$f : (V, p) \to (\mathbb{R}^k, 0), \ k = N - n,$$  \hspace{1cm} (3)

such that $X \cap V = f^{-1}(0)$ and such that $f$ is a submersion at all points of $X \cap V$, and in particular at $p$. Thus

$$df_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$$

is surjective, and hence the kernel of $df_p$ has dimension $n$. Our alternative definition of $T_pX$ is

$$T_pX = \text{kernel } df_p.$$  \hspace{1cm} (4)
The spaces (2) and (4) are both \( n \)-dimensional subspaces of \( T_p \mathbb{R}^N \), and we claim that these spaces are the same. (Notice that the definition (4) of \( T_p X \) doesn’t depend on \( \varphi \), so if we can show that these spaces are the same, the definitions (2) and (4) will depend neither on \( \varphi \) nor on \( f \).)

**Proof.** Since \( \varphi(U) \) is contained in \( X \cap V \) and \( X \cap V \) is contained in \( f^{-1}(0) \), \( f \circ \varphi = 0 \), so by the chain rule
\[
df_p \circ df_q = f(f \circ \varphi)_q = 0.
\]
Hence if \( v \in T_p \mathbb{R}^n \) and \( w = df_q(v), \ df_p(w) = 0 \). This shows that the space (2) is contained in the space (4). However, these two spaces are \( n \)-dimensional so the coincide.

\[\square\]

From the proof above one can extract a slightly stronger result:

**Theorem 1.** Let \( W \) be an open subset of \( \mathbb{R}^\ell \) and \( h : (W,q) \rightarrow (\mathbb{R}^N,p) \) a \( C^\infty \) map. Suppose \( h(W) \) is contained in \( X \). Then the image of the map
\[
dh_q : T_q \mathbb{R}^\ell \rightarrow T_p \mathbb{R}^N
\]
is contained in \( T_p X \).

**Proof.** Let \( f \) be the map (3). We can assume without loss of generality that \( h(W) \) is contained in \( V \), and so, by assumption, \( h(W) \subseteq X \cap V \). Therefore, as above, \( f \circ h = 0 \), and hence \( dh_q(T_q \mathbb{R}^\ell) \) is contained in the kernel of \( df_p \).

\[\square\]

This result will enable us to define the derivative of a mapping between manifolds. Explicitly: Let \( X \) be a submanifold of \( \mathbb{R}^N \), \( Y \) a submanifold of \( \mathbb{R}^m \) and \( g : (X,p) \rightarrow (Y,y_0) \) a \( C^\infty \) map. By Theorem 1 there exists a neighborhood, \( \mathcal{O} \), of \( X \) in \( \mathbb{R}^N \) and a \( C^\infty \) map, \( \tilde{g} : \mathcal{O} \rightarrow \mathbb{R}^m \) extending to \( g \). We will define
\[
(dg_p) : T_p X \rightarrow T_{y_0} Y
\]
to be the restriction of the map
\[
(d\tilde{g})_p : T_p \mathbb{R}^N \rightarrow T_{y_0} \mathbb{R}^m
\]
to \( T_p X \). There are two obvious problems with this definition:

1. Is the space 
\[
(d\tilde{g}_p)(T_p X)
\]
contained in \( T_{y_0} Y \)?

2. Does the definition depend on \( \tilde{g} \)?
To show that the answer to 1. is yes and the answer to 2. is no, let
\[ \varphi : (U, x_0) \to (X \cap V, p) \]
be a parameterization of \( X \), and let \( h = \tilde{g} \circ \varphi \). Since \( \varphi(U) \subseteq X \), \( h(U) \subseteq Y \) and hence by Theorem 2
\[ dh_{x_0}(T_{x_0}\mathbb{R}^n) \subseteq T_{y_0}Y. \]
But by the chain rule
\[ dh_{x_0} = d\tilde{g}_p \circ d\varphi_{x_0}, \] (8)
so by (2)
\[ (d\tilde{g}_p)(T_pX) \subseteq T_pY \] (9)
and
\[ (d\tilde{g}_p)(T_pX) = (dh)_{x_0}(T_{x_0}\mathbb{R}^n) \] (10)
Thus the answer to 1. is yes, and since \( h = \tilde{g} \circ \varphi = g \circ \varphi \), the answer to 2. is no.

From (5) and (6) one easily deduces

**Theorem 2** (Chain rule for mappings between manifolds). Let \( Z \) be a submanifold of \( \mathbb{R}^\ell \) and \( \psi : (Y, y_0) \to (Z, z_0) \) a \( \mathcal{C}^\infty \) map. Then \( d\psi_q \circ dg_p = d(\psi \circ g)_p \).

**Problem set**

1. What is the tangent space to the quadric, \( x_n^2 = x_1^2 + \cdots + x_{n-1}^2 \), at the point, \((1, 0, \ldots, 0, 1)\)?

2. Show that the tangent space to the \((n - 1)\)-sphere, \( S^{n-1} \), at \( p \), is the space of vectors, \((p, v) \in T_p\mathbb{R}^n \) satisfying \( p \cdot v = 0 \).

3. Let \( f : \mathbb{R}^n \to \mathbb{R}^k \) be a \( \mathcal{C}^\infty \) map and let \( X = \text{graph} f \). What is the tangent space to \( X \) at \((a, f(a))\)?

4. Let \( \sigma : S^{n-1} \to S^{n-1} \) be the anti-podal map, \( \sigma(x) = -x \). What is the derivative of \( \sigma \) at \( p \in S^{n-1} \)?

5. Let \( X_i \subseteq \mathbb{R}^{N_i}, i = 1, 2 \), be an \( n_i \)-dimensional manifold and let \( p_i \in X_i \). Define \( X \) to be the Cartesian product
\[ X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \]
and let \( p = (p_1, p_2) \). Describe \( T_pX \) in terms of \( T_{p_1}X_1 \) and \( T_{p_2}X_2 \).
6. Let $X \subseteq \mathbb{R}^N$ be an $n$-dimensional manifold and $\varphi_i : U_i \rightarrow X \cap V_i, i = 1, 2$. From these two parameterizations one gets an overlap diagram

$$
\begin{aligned}
W_1 &\xrightarrow{\psi} W_2 \\
X \cap V &\xrightarrow{d_1} &\xrightarrow{d_2} \\
W_1 &\xrightarrow{\psi} W_2
\end{aligned}
$$

where $V = V_1 \cap V_2$, $W_i = \varphi_i^{-1}(X \cap V)$ and $\psi = \varphi_2^{-1} \circ \varphi_1$.

(a) Let $p \in X \cap V$ and let $q_i = \varphi_i^{-1}(p)$. Derive from the overlap diagram (10) an overlap diagram of linear maps

$$
\begin{aligned}
(d\varphi_1)_{q_1} &\xrightarrow{T_p \mathbb{R}^N} (d\varphi_2)_{q_2} \\
T_{q_1} \mathbb{R}^n &\xrightarrow{(d\psi)_{q_1}} T_{q_2} \mathbb{R}^n
\end{aligned}
$$

(b) Use overlap diagrams to give another proof that $T_p X$ is intrinsically defined.