The theory of manifolds Lecture 2

Let $X$ be a subset of $\mathbb{R}^N$, $Y$ a subset of $\mathbb{R}^n$ and $f : X \to Y$ a continuous map. We recall

**Definition 1.** $f$ is a $C^\infty$ map if for every $p \in X$, there exists a neighborhood, $U_p$, of $p$ in $\mathbb{R}^N$ and a $C^\infty$ map, $g_p : U_p \to \mathbb{R}^n$, which coincides with $f$ on $U_p \cap X$.

We also recall:

**Theorem 1** (Munkres, §16, #3). If $f : X \to Y$ is a $C^1$ map, there exists a neighborhood, $U$, of $X$ in $\mathbb{R}^N$ and a $C^\infty$ map, $g : U \to \mathbb{R}^n$ such that $g$ coincides with $f$ on $X$.

We will say that $f$ is a diffeomorphism if it is one-one and onto and $f$ and $f^{-1}$ are both diffeomorphisms. In particular if $Y$ is an open subset of $\mathbb{R}^n$, $X$ is a simple example of what we will call a manifold. More generally,

**Definition 2.** A subset, $X$, of $\mathbb{R}^N$ is an $n$-dimensional manifold if, for every $p \in X$, there exists a neighborhood, $V$, of $p$ in $\mathbb{R}^m$, an open subset, $U$, in $\mathbb{R}^n$, and a diffeomorphism $\varphi : U \to X \cap V$.

Thus $X$ is an $n$-dimensional manifold if, locally near every point $p$, $X$ “looks like” an open subset of $\mathbb{R}^n$.

We’ll now describe how manifolds come up in concrete applications. Let $U$ be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $C^\infty$ map.

**Definition 3.** A point, $a \in \mathbb{R}^k$, is a regular value of $f$ if for every point, $p \in f^{-1}(a)$, $f$ is a submersion at $p$.

Note that for $f$ to be a submersion at $p$, $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ has to be onto, and hence $k$ has to be less than or equal to $N$. Therefore this notion of “regular value” is interesting only if $N \geq k$.

**Theorem 2.** Let $N - k = n$. If $a$ is a regular value of $f$, the set, $X = f^{-1}(a)$, is an $n$-dimensional manifold.

**Proof.** Replacing $f$ by $\tau_a \circ f$ we can assume without loss of generality that $a = 0$. Let $p \in f^{-1}(0)$. Since $f$ is a submersion at $p$, the canonical submersion theorem tells us that there exists a neighborhood, $O$, of $0$ in $\mathbb{R}^N$, a neighborhood, $U_0$, of $p$ in $U$ and a diffeomorphism, $g : O \to U_0$ such that

$$f \circ g = \pi$$  \hspace{1cm} (1)
where $\pi$ is the projection map

$$\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad (x, y) \rightarrow x.$$ 

Hence $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n = \mathbb{R}^n$ and by (1), $g$ maps $\mathcal{O} \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. However, $\mathcal{O} \cap \pi^{-1}(0)$ is a neighborhood, $V$, of 0 in $\mathbb{R}^n$ and $U_0 \cap f^{-1}(0)$ is a neighborhood of $p$ in $X$, and, as remarked, these two neighborhoods are diffeomorphic.

Some examples:

1. *The n-sphere.* Let

$$f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$$

be the map,

$$(x_1, \ldots, x_{n+1}) \rightarrow x_1^2 + \cdots + x_{n+1}^2 - 1.$$ 

Then

$$Df(x) = 2(x_1, \ldots, x_{n+1})$$

so, if $x \neq 0$ $f$ is a submersion at $x$. In particular $f$ is a submersion at all points, $x$, on the $n$-sphere

$$S^n = f^{-1}(0)$$

so the $n$-sphere is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$.

2. *Graphs.* Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a $C^1$ map and let

$$X = \text{graph } g = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k, \quad y = g(x)\}.$$ 

We claim that $X$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$.

**Proof.** Let

$$f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^k$$

be the map, $f(x, y) = y - g(x)$. Then

$$Df(x, y) = [-Dg(x), I_k]$$

where $I_k$ is the identity map of $\mathbb{R}^k$ onto itself. This map is always of rank $k$. Hence graph $g = f^{-1}(0)$ is an $n$-dimensional submanifold of $R^{n+k}$.

3. Munkres, §24, #6. Let $\mathcal{M}_n$ be the set of all $n \times n$ matrices and let $\mathcal{S}_n$ be the set of all symmetric $n \times n$ matrices, i.e., the set

$$\mathcal{S}_n = \{A \in \mathcal{M}_n, \ A = A^t\}.$$
The map 
\[
[a_{i,j}] \to (a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots)
\]
gives us an identification
\[
\mathcal{M}_n \cong \mathbb{R}^{n^2}
\]
and the map 
\[
[a_{i,j}] \to (a_{11}, \ldots a_{1n}, a_{22}, \ldots a_{2n}, a_{33}, \ldots a_{3n}, \ldots)
\]
gives us an identification
\[
\mathcal{S}_n \cong \mathbb{R}^{\frac{n(n+1)}{2}}.
\]
(Note that if \(A\) is a symmetric matrix, 
\[
a_{12} = a_{21}, a_{13} = a_{31}, a_{32} = a_{23}, \text{ etc.}
\]
so this map avoids redundancies.) Let 
\[
O(n) = \{A \in \mathcal{M}_n, A^tA = I\}.
\]
This is the set of orthogonal \(n \times n\) matrices, and the exercise in Munkres requires you to show that it’s an \(n(n-1)/2\)-dimensional manifold.

**Hint:** Let \( f : \mathcal{M}_n \to \mathcal{S}_n \) be the map \( f(A) = A^tA - I \). Then 
\[
O(n) = f^{-1}(0).
\]

These examples show that lots of interesting manifolds arise as zero sets of submersions, \( f : U \to \mathbb{R}^k \). We’ll conclude this lecture by showing that locally every manifold arises this way. More explicitly let \( X \subseteq \mathbb{R}^N \) be an \(n\)-dimensional manifold, \( p \) a point of \( X \), \( U \) a neighborhood of 0 in \( \mathbb{R}^n \), \( V \) a neighborhood of \( p \) in \( \mathbb{R}^N \) and \( \varphi : (U, 0) \to (V \cap X, p) \) a diffeomorphism. We will for the moment think of \( \varphi \) as a \(C^\infty\) map \( \varphi : U \to \mathbb{R}^N \) whose image happens to lie in \( X \).

**Lemma 3.** The linear map 
\[
D\varphi(0) : \mathbb{R}^n \to \mathbb{R}^N
\]
is injective.

**Proof.** \( \varphi^{-1} : V \cap X \to U \) is a diffeomorphism, so, shrinking \( V \) if necessary, we can assume that there exists a \(C^\infty\) map \( \psi : V \to U \) which coincides with \( \varphi^{-1} \) on \( V \cap X \). Since \( \varphi \) maps \( U \) onto \( V \cap X \), \( \psi \circ \varphi = \varphi^{-1} \circ \varphi \) is the identity map on \( U \). Therefore,
\[
D(\psi \circ \varphi)(0) = (D\psi)(p)D\varphi(0) = I
\]
by the change rule, and hence if \( D\varphi(0)v = 0 \), it follows from this identity that \( v = 0 \). \( \square \)
Lemma 6 says that \( \varphi \) is an immersion at 0, so by the canonical immersion theorem there exists a neighborhood, \( U_0 \), of 0 in \( U \) a neighborhood, \( V_p \), of \( p \) in \( V \), a neighborhood, \( \mathcal{O} \), of 0 in \( \mathbb{R}^N \) and a diffeomorphism

\[
g : (V_p, p) \rightarrow (\mathcal{O}, 0)
\]
such that

\[
i^{-1}(\mathcal{O}) = U_0 \tag{2}
\]
and

\[
g \circ \varphi = i, \tag{3}
\]
i being, as in lecture 1, the canonical immersion

\[
i(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0). \tag{4}
\]

By (3) \( g \) maps \( \varphi(U_0) \) diffeomorphically onto \( \varphi(U_0) \). However, by (2) and (3) \( i(U_0) \) is the subset of \( \mathcal{O} \) defined by the equations, \( x_i = 0, i = n + 1, \ldots, N \). Hence if \( g = (g_1, \ldots, g_N) \) the set, \( \varphi(U_0) = V_p \cap X \) is defined by the equations

\[
g_i = 0, \quad i = n + 1, \ldots, N. \tag{5}
\]

Let \( \ell = N - n \), let

\[
\pi : \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^\ell \rightarrow \mathbb{R}^\ell
\]
be the canonical submersion,

\[
\pi(x_1, \ldots, x_N) = (x_{n+1}, \ldots, x_N)
\]
and let \( f = \pi \circ g \). Since \( g \) is a diffeomorphism, \( f \) is a submersion and (5) can be interpreted as saying that

\[
V_p \cap X = f^{-1}(0). \tag{6}
\]

A nice way of thinking about Theorem 2 is in terms of the coordinates of the mapping, \( f \). More specifically if \( f = (f_1, \ldots, f_k) \) we can think of \( f^{-1}(a) \) as being the set of solutions of the system of equations

\[
f_i(x) = a_i, \quad i = 1, \ldots, k \tag{7}
\]
and the condition that \( a \) be a regular value of \( f \) can be interpreted as saying that for every solution, \( p \), of this system of equations the vectors

\[
(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(0) dx_j \tag{8}
\]
in \( T_p^*\mathbb{R}^n \) are linearly independent, i.e., the system (7) is an “independent system of defining equations” for \( X \).

**Problem set**

1. Show that the set of solutions of the system of equations
   \[ x_1^2 + \cdots + x_n^2 = 1 \]
   and
   \[ x_1 + \cdots + x_n = 0 \]
   is an \( n - 2 \)-dimensional submanifold of \( \mathbb{R}^n \).

2. Let \( S^{n-1} \) be the \( n \)-sphere in \( \mathbb{R}^n \) and let
   \[ X_a = \{ x \in S^{n-1}, \ x_1 + \cdots + x_n = 0 \}. \]
   For what values of \( a \) is \( X_a \) an \( (n - 2) \)-dimensional submanifold of \( S^{n-1} \)?

3. Show that if \( X_i, \ i = 1, 2, \) is an \( n_i \)-dimensional submanifold of \( \mathbb{R}^{N_i} \) then
   \[ X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \]
   is an \( (n_1 + n_2) \)-dimensional submanifold of \( \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \).

4. Show that the set
   \[ X = \{(x, v) \in S^{n-1} \times \mathbb{R}^n, \ x \cdot v = 0\} \]
   is a \( 2n - 2 \)-dimensional submanifold of \( \mathbb{R}^n \times \mathbb{R}^n \). (Here “\( x \cdot v \)” is the dot product, \( \sum x_i v_i \).)

5. Let \( g : \mathbb{R}^n \rightarrow \mathbb{R}^k \) be a \( C^\infty \) map and let \( X = \text{graph} \ g \). Prove directly that \( X \) is an \( n \)-dimensional manifold by proving that the map
   \[ \gamma : \mathbb{R}^n \rightarrow X, \quad x \rightarrow (x, g(x)) \]
   is a diffeomorphism.