6 Integration on manifolds, Lecture 6

To formulate the divergence theorem we need one final ingredient: Let D be an open subset of X and $\bar{D}$ its closure.

Definition 1. *D is a* smooth domain *if*

(a) its boundary is an $(n-1)$-dimensional submanifold of X and

(b) the boundary of D coincides with the boundary of $\bar{D}$.

Examples.

1. The n-ball, $x_1^2 + \cdots + x_n^2 < 1$, whose boundary is the sphere, $x_1^2 + \cdots + x_n^2 = 1$.
2. The n-dimensional annulus,
$$1 < x_1^2 + \cdots + x_n^2 < 2$$
whose boundary consists of the spheres,
$$x_1^2 + \cdots + x_n^2 = 1 \text{ and } x_1^2 + \cdots + x_n^2 = 2\,.$$
3. Let S^{n-1} be the unit sphere, $x_1^2 + \cdots + x_2^2 = 1$ and let $D = \mathbb{R}^n - S^{n-1}$. Then the boundary of D is S^{n-1} but D is not a smooth domain since the boundary of $\bar{D}$ is empty.

The simplest example of a smooth domain is the half-space (5.15). We will show that every bounded domain looks locally like this example.

Theorem 1. *Let D be a smooth domain and p a boundary point of D. Then there exists a neighborhood, U, of p in X, an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\varphi_0 : U_0 \to U$ such that φ_0 maps $U_0 \cap \mathbb{H}^n$ onto $U \cap D$.*

Proof. Let Z be the boundary of D. Then by Theorem 5.5 there exists a neighborhood, U of p in X, an open ball, U_0 in $\mathbb{R}^n$, with center at $q \in Bd\mathbb{H}^n$, and a diffeomorphism,
$$\varphi : (U_0, q) \to (U, p)$$
mapping $U_0 \cap Bd\mathbb{H}^n$ onto $U \cap Z$. Thus for $\varphi^{-1}(U \cap D)$ there are three possibilities.

i. $\varphi^{-1}(U \cap D) = (\mathbb{R}^n - Bd\mathbb{H}^n) \cap U_0$.

ii. $\varphi^{-1}(U \cap D) = \mathbb{H}^n \cap U_0$.

or

iii. $\varphi^{-1}(U \cap D) = \mathbb{H}^n U_0$.

However, i. is excluded by the second hypothesis in Definition 1 and if ii. occurs we can rectify the situation by composing φ with the map, $(x_1, \ldots, x_n) \to (-x_n, x_2, \ldots, x_n)$.

$\square$

Definition 2. *We will call an open set, U, with the properties above a D-adapted parametrizable open set.*

We will leave the following as an exercise:

Proposition 2. *The boundary of D is two-sided.*

Hint: At boundary points, p, of D there are two kinds of non-tangential vector fields: inward-pointing vector fields and outward-pointing vector fields. In the divergence theorem we will co-orient the boundary of D by giving it the *outward-pointing* co-orientation.

Theorem 3 (Divergence Theorem)**.** *If v is a vector field on X and σ a compactly supported $\mathcal{C}^\infty$ density the flux of (v, σ) through the boundary of D is equal to the integral over D of $L_v\sigma$.*

The key ingredient of the proof of this theorem is the following lemma.

Lemma 4. *Let X_i, $i = 1, 2$, be an n-dimensional manifold and $D_i \subseteq X_i$ a smooth domain. If (X_1, D_1) is diffeomorphic to (X_2, D_2) then the divergence theorem is true for (X_1, D_1) if and only if it is true for (X_2, D_2).*

Proof. This follows from the identities (4.18) and (5.13) and the global change of variables formula for integrals of densities that we proved in Lecture 3.

Let's now prove the theorem itself. By a partition of unity argument we can assume one of the following three alternatives holds.

1. σ is supported in the exterior of D.
2. σ is supported in D.
3. σ is supported in a D-adapted parametrizable open set of U.

In case 1 there is nothing to prove. The integral of $L_v\sigma$ over D and the integral of σ_v over the boundary are both zero. In case 2

$$\int_D L_v\sigma = \int_X L_v\sigma$$

and since σ is zero on the boundary the flux through the boundary is zero, so in this case Theorem 3 follows from Theorem 4.3. Let's prove the theorem in case 3. By Theorem 1 there exists an open ball, U_0, in $\mathbb{R}^n$ and a diffeomorphism of U_0 onto U

mapping $U_0 \cap \mathbb{H}^n$ onto $U \cap D$, hence by Lemma 4 it suffices to prove the theorem for $\mathbb{H}^n$ and $\mathbb{R}^n$. Let's do so. Let v be the vector field

$$\sum a_i \frac{\partial}{\partial x_i}$$

and σ the density

$$\sigma = \varphi \sigma_{\text{Leb}}$$

with $\varphi \in \mathcal{C}_0^\infty(U_0)$. Let $v_1 = \partial/\partial x_1$. Then

$$\sigma_v = (a_1\varphi)(0, x_2, \ldots, x_n)\iota\left(\frac{\partial}{\partial x_i}\right)\sigma_{\text{Leb}}\,.$$

However, by (1.20) $\iota(\partial/\partial x_i)\sigma_{\text{Leb}}$ is the Lebesgue density on $\mathbb{R}^{n-1}$, so

$$\text{Flux}(v, \sigma) = \int_{\mathbb{R}^{n-1}} \varphi(0, x_2, \ldots, x_n) a_1(0, x_2, \ldots, x_n)\, dx_2 \cdots dx_n\,. \tag{6.1}$$

On the other hand by (4.8)

$$L_v\sigma = \sum \left(\frac{\partial}{\partial x_i} a_i\varphi\right)\sigma_{\text{Leb}}$$

so the right hand side of the divergence formula is the sum from 1 to n of the integrals

$$\int_{\mathbb{H}^n} \left(\frac{\partial}{\partial x_i} a_i\varphi\right) dx\,. \tag{6.2}$$

By Fubini's theorem we can write this integral as an iterated integral, integrating first with respect to the variable, x_i, then with respect to the other variables. For $i \neq 1$ the integration with respect to x_i is over the interval, $-\infty < x_i < \infty$, so we get

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x_i}(a_i\varphi)\, dx_i = 0$$

since $a_i\varphi$ is compactly supported in the variable, x_i. On the other hand for $i = 1$, the integration is over the integral, $-\infty < x_1 < 0$ so we get

$$\int_{-\infty}^{0} \frac{\partial}{\partial x_i}(a_1\varphi)\, dx_1 = a_1(0, x_2, \ldots, x_n)\varphi(0, x_1, \ldots, x_n)\,,$$

and integrating over the remaining variables we get (6.1).

□

Exercises.

1. Let B^n be the unit ball in $\mathbb{R}^n$ and S^{n-1} the unit $(n-1)$-sphere. Prove that
$$\mathrm{vol}\,(S^{n-1}) = n\,\mathrm{vol}\,(B^n)\,.$$
Hint: Lecture 5, exercise 3.

2. Let D be the annulus,
$$a < x_1^2 + \cdots + x_2^2 < b\,, \quad 0 < a < b\,.$$
From the divergence theorem conclude that if v is a divergence free vector field, the flux of $(v, \sigma_{\mathrm{Leb}})$ through the sphere, $x_1^2 + \cdots + x_n^2 = a$ is equal to its flux through the sphere, $x_1^2 + \cdots + x_n^2 = b$.

3. Let U be a bounded open subset of $\mathbb{R}^{n-1}$ and let $X = U \times \mathbb{R}$. Given a positive $\mathcal{C}^\infty$ function $f : U \to \mathbb{R}$ let D be the open subset of X defined by
$$0 < x_n < f(x_1, \ldots, x_{n-1})\,.$$
For $v = x_n \partial/\partial x_n$ and $\sigma = \sigma_{\mathrm{Leb}}$, verify the divergence theorem by computing the flux of (v, σ) through the boundary of D and the integral of the divergence of v over D and showing they're equal.

4. Let D be a bounded smooth domain in $\mathbb{R}^n$ and v a vector field. The classical divergence theorem of multivariable calculus asserts that
$$\int_D \mathrm{div}\,(v) = \int_{BdD} (n \cdot v)\sigma_{\mathrm{Leb}}$$
where, at $p \in BdD$, n_p is the unit outward normal vector and $(n \cdot v)(p)$ is the dot product of $v(p)$ and n_p. Deduce this version of the divergence theorem from the divergence theorem that we proved above.

5. Let v be a vector field on $\mathbb{R}^n$ and for $a \in \mathbb{R}^n$ let Δ be the n-cube
$$-\epsilon + a_i < x_i < \epsilon + a_i\,, \quad i = 1, \ldots, n\,.$$
Prove that if $\mathrm{Flux}(v, \Delta)$ is the sums of the fluxes of $(v, \sigma_{\mathrm{Leb}})$ over the $2n$ faces of Δ then for ϵ small,
$$\mathrm{Flux}(v, \Delta) \doteq \mathrm{div}\,(v)(a)\,\mathrm{vol}\,(\Delta)\,.$$

6. Prove Proposition 2.

Hint: Let $\mathbb{U} = \{U_i\,,\ i = 1, 2, \ldots\}$ be a covering of the boundary of D by D-adapted parametrizable open sets. Let U be their union and $\rho_i \in \mathcal{C}_0^\infty(U)$, $i = 1, 2, \ldots$, a partition of unity subordinate to $\mathbb{U}$. Show that on each U_i there exists and outward-pointing vector field, v_i, and let $v = \sum \rho_i v_i$.