# 4 Integration on manifolds, Lecture 4

In the next three lectures of this course we will prove a manifold version of one of the fundamental theorems in multi-variable calculus: the divergence theorem. The calculus version of this theorem says that if $S \subseteq \mathbb{R}^3$ is a closed surface and $v$ a vector field, the flux of $v$ through $S$ is equal to integral of the divergence of $v$ over the region bounded by $S$. To extend this theorem to manifolds we will need manifold versions of the notion of divergence and flux. The notion of divergence is closely related to another important manifold notion: *the Lie derivative of a density by a vector field*. We'll discuss both of these concepts below, and discuss the concept of flux in Lecture 5.

Let $X$ be an $n$-dimensional manifold and $v$ a vector field on $X$. To simplify slightly the exposition in what follows we'll assume that $v$ is complete, and hence that it generates a one-parameter group of diffeomorphisms

$$f_t : X \to X\,, \quad -\infty < t < \infty\,. \tag{4.1}$$

Let's recall that if $\varphi$ is a $\mathcal{C}^\infty$ function on $X$ its Lie derivative with respect to $v$ can be defined by the formula

$$L_v\varphi = \left(\frac{d}{dt} f_t^*\varphi\right)(t=0)\,. \tag{4.2}$$

This formula makes sense for densities as well. Namely if $\sigma$ is an element of $\mathcal{D}^\infty(X)$ we can define its Lie derivative by the recipe:

$$L_v\sigma = \left(\frac{d}{dt} f_t^*\sigma\right)(t=0) \quad (*)\,. \tag{4.3}$$

Moreover, the operations (4.2) and (4.3) are compatible: if $\varphi$ is a $\mathcal{C}^\infty$ function and $\sigma$ a $\mathcal{C}^\infty$ density then[1]

$$f_t^*\varphi\sigma \;=\; (f_t^*\varphi)f_t^*\sigma\,,$$

hence

[1]This definition makes sense without the assumption that $v$ be complete, however it is slightly more complicated. In the vector field segment of this course we pointed out that for every point, $p$, in $X$ there exists a neighborhood, $U$, of $p$, an interval, $-\epsilon < t < \epsilon$, and a family of local diffeomorphisms

$$f_t : U \to X$$

such that for $q \in U$ the curve, $\delta_q(t) = f_t(q)$, $-\epsilon < t < \epsilon$, is an integral curve of $v$ with initial point, $\gamma_q(0) = q$. In other words, $v$ generates a "local" one-parameter group of diffeomorphisms on $U$, so the Lie derivative of $\sigma$ with respect to $v$ can still be defined by the recipe (4.3) in the vicinity of $p$ for every $p \in X$.

$$\frac{d}{dt}f_t^*(\varphi\sigma) \quad = \quad \frac{d}{dt}f_t^*\varphi f_t^*\sigma + f_t^*\sigma + f_t^*\varphi\frac{d}{dt}f_t^*\sigma$$

which for $t = 0$ reduces to:

$$L_v(\varphi\sigma) = (L_v\varphi)\sigma + \varphi L_v\sigma\,. \tag{4.4}$$

To see what this Lie differentiation operation looks like "locally" let's compute (4.3) for the special case of open subsets of $\mathbb{R}^n$. We'll begin by proving a linear algebra lemma which we'll need for this computation.

**Lemma 1.** *Let $A(t) = [a_{i,j}(t)]$, $-\epsilon < t < \epsilon$, be an $n \times n$ matrix whose entries, $a_{i,j}(t)$, are $\mathcal{C}^\infty$ functions of $t$. Then if $A(0)$ is the identity matrix*

$$\frac{d}{dt}(\det A)(0) \quad = \quad \mathrm{trace}\frac{d}{dt}A(0) \tag{4.5}$$

*where*

$$\mathrm{trace}\frac{d}{dt}A(0) \quad = \quad \sum_{i=1}^{n}\frac{d}{dt}a_{i,i}(0)\,. \tag{4.6}$$

*Proof.* By Theorem 2.15 in Munkres, §2

$$\det A(t) = \sum_{i=1}^{n}(-1)^i a_{1,i}(t)\det A_{1,i}(t)$$

where $A_{1,i}(t)$ is the $(n-1)\times(n-1)$ matrix obtained by deleting from $A(t)$ its first row and $i^{\text{th}}$ column. Thus $\frac{d}{dt}\det A(t)$ is equal at $t = 0$ to the sum of

$$\sum(-1)^i\frac{d}{dt}a_{1,i}(0)\det A_{1,i}(0) \tag{4.7}$$

and

$$\sum(-1)^i a_{1,i}(0)\frac{d}{dt}\det A_{1,i}(0)\,. \tag{4.8}$$

However, $A_{1,1}(0)$ is the identity $(n-1)\times(n-1)$ matrix and for $i \neq 1$, the first column of the matrix $A_{1,i}(0)$ consists entirely of zeros. Thus $\det A_{1,1}(0) = 1$ and, for $i = 1$, $\det A_{1,i}(0) = 0$, so (4.7) is just $\frac{d}{dt}a_{1,1}(0)$. Moreover, $a_{1,1}(0) = 1$ and $a_{1,i}(0) = 0$ for $i \neq 1$, so (4.8) is just $\frac{d}{dt}A_{1,1}(0)$. Arguing by induction we can assume that the theorem is true for $n-1$ and hence that (4.8) is equal to

$$\sum_{i=2}^{n}\frac{d}{dt}a_{i,i}(0)\,.$$

Adding to this the term (4.7), which we've just observed to be $\frac{d}{dt}a_{1,1}(0)$, we get the formula (4.5).

□

Now let $U$ be an open subset of $\mathbb{R}^n$ and let $v$ be the vector field

$$v = \sum v_i \frac{\partial}{\partial x_i}\,, \tag{4.9}$$

A density, $\sigma \in \mathcal{D}^\infty(U)$ can be written as a product $\varphi\sigma_{\text{Leb}}$ with $\varphi$ in $\mathcal{C}^\infty(U)$, so by (4.4)

$$L_v\sigma = ((L_v\varphi)\sigma_{\text{Leb}} + \varphi L_v\sigma_{\text{Leb}}\,, \tag{4.10}$$

so to compute $L_v\sigma$ it suffices to compute $L_v\sigma_{\text{Leb}}$. Let

$$f_t(x) = (f_1(x,t), \cdots, f_n(x,t))$$

then by (2.4):

$$f_t^*\sigma_{\text{Leb}} = |\det J(t)|\sigma_{\text{Leb}} \tag{4.11}$$

where

$$J(t) = \left[\frac{\partial f_i}{\partial x_j}(x,t)\right]\,.$$

Note that since $f_0$ is the identity map, $J(0)$ is the identity matrix, so its determinant is 1. Moreover, $J(t)$ is an invertible matrix, so its determinant is non-zero. Hence since $\det J(t)$ depends continuously on $t$ it has to be positive for all $t$, so we can drop the absolute value sign from (4.11) and write (4.11) in the form

$$f_t^*\sigma_{\text{Leb}} = \det J(t)\sigma_{\text{Leb}}\,.$$

Thus

$$L_v\sigma_{\text{Leb}} = \left(\frac{d}{dt}f_t^*\sigma_{\text{Leb}}\right)(t=0) = \frac{d}{dt}(\det J)(0)\sigma_{\text{Leb}}\,,$$

and hence by the lemma:

$$L_v\sigma_{\text{Leb}} = \left(\sum_{i=1}^n \frac{d}{dt}\frac{\partial f_i}{\partial x_i}(x,0)\right)\sigma_{\text{Leb}}\,. \tag{4.12}$$

Now recall that $f_t(x) = \gamma_x(t)$ where $\gamma_x(t)$ is the integral curve of $v$ with initial point $\gamma_x(0) = x$, i.e,

$$\frac{d}{dt}\gamma_x(0) = v(x)$$

and hence

$$\frac{d}{dt}f_i(x,0) = v_i(x)\,.$$

Differentiating this identity with respect to $x_i$ we get

$$\frac{d}{dt}\frac{\partial f_i}{\partial x_i}(x,0) = \sum \frac{\partial v_i(x)}{\partial x_i} \tag{4.13}$$

and hence, finally, by (4.12)

$$L_v \sigma_{\text{Leb}} = \text{div}\,(v)\sigma_{\text{Leb}} \tag{4.14}$$

where

$$\text{div}\,(v) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} \tag{4.15}$$

is the *divergence* of $v$. Coming back to (4.10) we get for $L_v\sigma$ the formula

$$\left( \sum v_i \frac{\partial \varphi}{\partial x_i} + \varphi \sum \frac{\partial v_i}{\partial x_i} \sigma_{\text{Leb}} \right)$$

so we can write (4.10) in the more compact form:

$$L_v \sigma = \left( \sum \frac{\partial}{\partial x_i}(v_i \varphi) \right) \sigma_{\text{Leb}}\,. \tag{4.16}$$

We will next show how the Lie differentiation operation behaves under global change of variables. Let $X$ and $Y$ be $n$-dimensional manifolds and $\gamma : X \to Y$ a diffeomorphism. In the "theory of manifolds" segment of this course we showed that if $v$ is a vector field on $X$ and $w = \gamma_* v$ then for $\varphi \in \mathcal{C}^\infty(Y)$

$$L_v \gamma^* \varphi = \gamma^* L_w \varphi\,. \tag{4.17}$$

We will prove that the same identity holds for densities, i.e., for $\sigma$ in $\mathcal{D}^\infty(Y)$

$$L_v \gamma^* \sigma = \gamma^* L_w \sigma\,. \tag{4.18}$$

*Proof.* Let $f_t : X \to X$ be the one-parameter group of diffeomorphisms generated by $v$. Then the one-parameter group of diffeomorphisms generated by $w$ is the group

$$g_t = \gamma \circ f_t \circ \gamma^{-1}$$

so

$$\begin{aligned} g_t \circ \gamma &= \gamma \circ f_t \qquad \text{and hence} \\ \gamma^* g_t^* \sigma &= f_t^* \gamma^* \sigma\,. \end{aligned}$$

Differentiating this identity with respect to $t$ and setting $t = 0$ we get (4.15).

□

One application of this change of variables formula is the following result (which we've implicitly been assuming to be true, but nonetheless requires a proof).

**Theorem 2.** *If $\sigma$ is in $\mathcal{D}^\infty(X)$, so is $L_v\sigma$.*

*Proof.* One has to prove that $L_v\sigma$ is $\mathcal{C}^\infty$ on parametrizable open subset of $X$ and hence, by Theorem 2, that $L_v\sigma$ is $\mathcal{C}^\infty$ when $X$ is an open subset of $\mathbb{R}^n$. This, however, is obvious by the formula (4.15).

□

The statement and proof of the divergence theorem requires some further machinery (which we'll develop in the next lecture) but we can already prove an important special case of this theorem.

**Theorem 3.** *If $\sigma$ is in $\mathcal{D}^\infty(X)$*

$$\int_X L_v\sigma = 0\,. \tag{4.19}$$

*Proof.* Let $f_t : X \to X$ $-\infty < t < \infty$ be the one-parameter group of diffeomorphisms generated by $v$. Then by the global change of variables formula for integration which we proved in Lecture 3

$$\int_X f_t^*\sigma = \int_X \sigma$$

and hence

$$0 = \frac{d}{dt}\int_X f_t^*\sigma = \int_X \frac{d}{dt} f_t^*\sigma$$

and at $t = 0$ the term on the right is $\int_X L_v\sigma$ .

□

We still have to show how to extend the notion of divergence to manifolds. This we will do as follows: If $v$ is a vector field on $X$ then, as we observed in Lecture 2, we can write the $\mathcal{C}^\infty$ density, $L_v\sigma_{\text{vol}}$, as the product of a $\mathcal{C}^\infty$ function with $\sigma_{\text{vol}}$, and we'll call this $\mathcal{C}^\infty$ function the *divergence* of $v$, i.e., we will define the divergence of $v$ by the identity

$$L_v\sigma_{\text{vol}} = \text{div}\,(v)\sigma_{\text{vol}}\,. \tag{4.20}$$

For $\mathbb{R}^n$ $\sigma_{\text{vol}} = \sigma_{\text{Leb}}$, so for vector fields on $\mathbb{R}^n$ this definition coincides with the calculus definition (4.15) of divergence.

**Exercises.**

1. For $X = \mathbb{R}^n$ derive Theorem 2 directly from the formula (4.15).

2. A vector field, $v$, on $\mathbb{R}^n$ is divergence-free if $\text{div}\,(v) = 0$. Show that the vector fields below are divergence free.

   (a) The coordinate vector fields, $\partial/\partial x_i$.

   (b) The vector field

$$v = \sum x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\,. \tag{4.21}$$

(c) The vector field

$$v = (x_1^2 + \cdots + x_n^2)^{-n/2} \sum x_i \frac{\partial}{\partial x_i} \,. \tag{4.22}$$

3. Let $[a_{i,j}(x)]$ be a skew-symmetric $n \times n$ matrix of functions, $a_{i,j} \in \mathcal{C}^\infty(\mathbb{R}^n)$, i.e.,

$$a_{i,j} = a_{j,i} \,.$$

Show that the vector field

$$v = \sum \left( \frac{\partial}{\partial x_i} a_{i,j} \right) \frac{\partial}{\partial x_j} \tag{4.23}$$

is divergence-free.

4. Let $v$ be a vector field on $\mathbb{R}^n$. Show that there exists a vector field of the form

$$w = f \frac{\partial}{\partial x_n} \,, \quad f \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{4.24}$$

with the property $\operatorname{div}(v) = \operatorname{div}(w)$.

5*. Prove by induction on $n$ that every divergence-free vector field on $\mathbb{R}^n$, $n > 1$, is of the form (4.23).

6.

(a) Let $X \subseteq \mathbb{R}^n$ be the $(n-1)$-sphere. Show that the vector field

$$\sum x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$$

is tangent to $X$ at all points, $p \in X$ and hence restricts to a vector field, $v$, on $X$.

(b) Prove that the divergence of this vector field is zero.