3 Integration on manifolds, Lecture 3

In this section we will show how to integrate densities over manifolds. First, however, we will have to explain how to integrate densities over open subsets, \( U \), of \( \mathbb{R}^n \). Recall that if \( \varphi \) is in \( \mathcal{D}^\infty(U) \) it can be written as a product, \( \sigma = \psi \sigma_{\text{Leb}} \), where \( \psi \) is in \( \mathcal{C}^\infty(U) \). We will say that \( \sigma \) is integrable over \( U \) if \( \psi \) is integrable over \( U \), and will define the integral of \( \sigma \) over \( U \) to be the usual Riemann integral

\[
\int_U \sigma = \int_U \psi \, dx. \tag{3.1}
\]

The advantage of using “density” notation for this integral is that it makes the change of variables formula more transparent. Namely if \( U_1 \) is an open subset of \( \mathbb{R}^n \) and \( f : U_1 \to U \) a diffeomorphism, then by (2.4) \( f^* \sigma = \psi_1 \sigma_{\text{Leb}} \) where

\[
\psi_1 = \psi \circ f \left| \det \left[ \frac{\partial f_i}{\partial x_j} \right] \right|
\]

and hence by the change of variables formula\(^1\) \( \psi_1 \) is integrable over \( U_1 \) and

\[
\int_{U_1} \psi_1 \, dx = \int_U \psi \, dx.
\]

Thus using density notation the change of variables formula takes the much simpler form

\[
\int_{U_1} f^* \sigma = \int_U \sigma. \tag{3.3}
\]

Now let \( X \subseteq \mathbb{R}^N \) be an \( n \)-dimensional manifold. Our goal below will be to define the integral

\[
\int_W \sigma \tag{3.4}
\]

where \( W \) is an open subset of \( X \) and \( \sigma \) is a compactly supported \( \mathcal{C}^\infty \) density. We’ll first show how to define this integral when the support of \( \sigma \) is contained in a “parametrizable” open subset of \( X \) and then, using partition of unity argument, define it in general.

**Definition 1.** An open subset, \( U \), of \( X \) is parametrizable if there exists an open set, \( U_0 \), in \( \mathbb{R}^n \) and a diffeomorphism, \( \varphi_0 : U_0 \to U \).

In other words “\( U \) is parametrizable” means that there exists a parameterization, \((U_0, \varphi_0)\), of \( U \). It’s clear that if \( U \) is parametrizable every open subset of \( U \) is parametrizable, and, in particular, if \( U_1 \) and \( U_2 \) are parametrizable, so is \( U_1 \cap U_2 \). Moreover the definition of manifold says that every point, \( p \in X \), is contained in a parametrizable open set.

\(^1\)See Theorem 17.2 in Munkres.
Let \( \sigma \) be an element of \( \mathcal{D}_0^\infty(X) \) whose support is contained in a parametrizable open set \( U \). Picking a parameterization, \( \varphi_0 : U_0 \to U \) we will define the integral of \( \sigma \) over \( W \) by defining it to be
\[
\int_W \sigma = \int_{W_0} \varphi_0^* \sigma
\]  
(3.5)
where \( W_0 = \varphi_0^{-1}(W) \). Note that since \( \sigma \) is compactly supported on \( U \), \( \varphi_0^* \sigma \) is a product, \( \varphi_0^* \sigma = \psi \sigma_{\text{Leb}} \), with \( \psi \) in \( \mathcal{C}_0^\infty(U_0) \). Hence by Munkres, Theorem 15.2, \( \psi \) is integrable over \( W_0 \) and hence so is \( \varphi^* \sigma \). We will prove

**Lemma 1.** The definition (3.5) doesn't depend on the choice of the parameterization, \((U_0, \varphi_0)\).

**Proof.** Let \((U_1, \varphi_1)\) be another parameterization of \( U \) and let \( f = \varphi_1^{-1} \circ \varphi_0 \). Since \( \varphi_0 \) and \( \varphi_1 \) are diffeomorphisms of \( U_0 \) and \( U_1 \) onto \( U \) \( f \) is a diffeomorphism of \( U_0 \) onto \( U_1 \) with the property
\[
\varphi_1 \circ f = \varphi_0.
\]  
(3.6)
In particular if \( W_i = \varphi_i^{-1}(W) \), \( i = 1, 2 \) it follows from (3.6) that \( f \) maps \( W_0 \) diffeomorphically onto \( W_1 \) and from the chain rule it follows that \( f^* \varphi_1^* \sigma = \varphi_0^* \sigma \). Hence by (3.3)
\[
\int_{W_0} \varphi_0^* \sigma = \int_{W_1} \varphi_1^* \sigma.
\]  
(3.7)
In other words (3.5) is unchanged if we substitute \((U_1, \varphi_1)\) for \((U_0, \varphi_0)\).

\[\square\]

From the additivity of the Riemann integral for integrable functions on open subsets of \( \mathbb{R}^n \) we also conclude

**Lemma 2.** If \( \sigma_i \in \mathcal{D}_0^\infty(X) \), \( i = 1, 2 \), is supported on \( U \)
\[
\int_W \sigma_1 + \sigma_2 = \int_W \sigma_1 + \int_W \sigma_2
\]
and if \( \sigma \in \mathcal{D}_0^\infty(X) \) is supported on \( U \) and \( c \in \mathbb{R} \)
\[
\int_W c \sigma = c \int_W \sigma.
\]

To define the integral (3.4) for arbitrary elements of \( \mathcal{D}^\infty(X) \) we will resort to the same partition of unity arguments that we used earlier in the course to define improper integrals of functions over open subsets of \( \mathbb{R}^n \). To do so we'll need the following manifold version of Munkres’ Theorem 16.3.

**Theorem 3.** Let
\[
U = \{U_\alpha \mid \alpha \in \mathcal{I}\}
\]  
(3.8)
be a covering of \( X \) be open subsets. Then there exists a family of functions, \( \rho_i \in \mathcal{C}_0^\infty(X) \), \( i = 1, 2, 3, \ldots \), with the properties
(a) \( \rho_i \geq 0 \).

(b) For every compact set, \( C \subseteq X \) there exists a positive integer \( N \) such that if \( i > N \), \( \text{supp } \rho_i \cap C = \emptyset \).

(c) \( \sum \rho_i = 1 \).

(d) For every \( i \) there exists an \( \alpha \in \mathcal{I} \) such that \( \text{supp } \rho_i \subseteq U_\alpha \).

Remark. Conditions (a)–(c) say that the \( \rho_i \)'s are a partition of unity and (d) says that this partition of unity is subordinate to the covering (3.8).

Proof. To simplify the proof a bit we’ll assume that \( X \) is a closed subset of \( \mathbb{R}^N \). For each \( U_\alpha \) choose an open subset, \( \mathcal{O}_\alpha \) in \( \mathbb{R}^N \) with

\[
U_\alpha = \mathcal{O}_\alpha \cap X
\]

and let \( \mathcal{O} \) be the union of the \( \mathcal{O}_\alpha \)'s. By the theorem in Munkres that we cited above there exists a partition of unity, \( \tilde{\rho}_i \in C_0^\infty(\mathcal{O}) \), \( i = 1, 2, \ldots \), subordinate to the covering of \( X \) by the \( \mathcal{O}_\alpha \)'s. Let \( \rho_i \) be the restriction of \( \tilde{\rho}_i \) to \( X \). Since the support of \( \tilde{\rho}_i \) is compact and \( X \) is closed, the support of \( \rho_i \) is compact, so \( \rho_i \in C_0^\infty(X) \) and it’s clear that the \( \rho_i \)'s inherit from the \( \tilde{\rho}_i \)'s the properties (a)–(d).

Now let the covering (3.8) be any covering of \( X \) by parametrizable open sets and let \( \rho_i \in C_0^\infty(X) \), \( i = 1, 2, \ldots \), be a partition of unity subordinate to this covering. Given \( \sigma \in D_0^\infty(X) \) we will define the integral of \( \sigma \) over \( W \) by the sum

\[
\sum_{i=1}^{\infty} \int_W \rho_i \sigma.
\]

Note that since each \( \rho_i \) is supported in some \( U_\alpha \) the individual summands in this sum are well-defined and since the support of \( \sigma \) is compact all but finitely many of these summands are zero by part (b) of Theorem 3. Hence the sum itself is well-defined. Let’s show that this sum doesn’t depend on the choice of \( \mathcal{U} \) and the \( \rho_i \)'s. Let \( \mathcal{U}' \) be another covering of \( X \) by parametrizable open sets and \( \rho'_j \), \( j = 1, 2, \ldots \), a partition of unity subordinate to \( \mathcal{U}' \). Then

\[
\sum_j \int_W \rho'_j \sigma = \sum_j \int_W \sum_i \rho'_j \rho_i \sigma = \sum_j \left( \sum_i \int_W \rho'_j \rho_i \sigma \right)
\]
by Lemma 2. Interchanging the orders of summation and resumming with respect to the \( j \)'s this sum becomes

\[
\sum_i \int_W \rho_j \rho_i \sigma
\]

or

\[
\sum_i \int_W \rho_i \sigma.
\]

Hence

\[
\sum_i \int_W \rho_j \sigma = \sum_i \int_W \rho_i \sigma,
\]

so the two sums are the same. Q.E.D.

From (3.10) and Lemma 2 one easily deduces

**Proposition 4.** For \( \sigma_i \in \mathcal{D}_0^\infty(X) \), \( i = 1, 2 \)

\[
\int_W \sigma_1 + \sigma_2 = \int_W \sigma_1 + \int_W \sigma_2\tag{3.12}
\]

and for \( \sigma \in \mathcal{D}_0^\infty(X) \) and \( c \in \mathbb{R} \)

\[
\int_W c\sigma = c \int_W \sigma\,.\tag{3.13}
\]

In the definition of the integral (3.4) we’ve allowed \( W \) to be an arbitrary open subset of \( X \) but required \( \sigma \in \mathcal{D}^\infty(X) \) to be compactly supported. This integral is also well-defined if we allow \( \sigma \) to be an arbitrary element of \( \mathcal{D}^\infty(X) \) but require the closure of \( W \) in \( X \) to be compact. To see this, note that under this assumption the sum (3.10) is still a finite sum, so the definition of the integral still makes sense, and the double sum on the right side of (3.11) is still a finite sum so it’s still true that the definition of the integral doesn’t depend on the choice of partitions of unity. In particular if the closure of \( W \) in \( X \) is compact we will define the volume of \( W \) to be the integral,

\[
\text{vol}(W) = \int_W \sigma_{\text{vol}}\,,\tag{3.14}
\]

and if \( X \) itself is compact we’ll define its volume to be the integral

\[
\text{vol}(X) = \int_X \sigma_{\text{vol}}\,.\tag{3.15}
\]

(For an alternative way of defining the volume of a manifold see Munkres, §22.)

We’ll conclude this discussion of integration by proving a manifold version of the change of variables formula (3.3).
**Theorem 5.** Let $X'$ and $X$ be $n$-dimensional manifolds and $f : X' \to X$ a diffeomorphism. If $W$ is an open subset of $X$ and $W' = f^{-1}(W)$

$$
\int_{W'} f^* \sigma = \int_W \sigma \quad (3.16)
$$

for all $\sigma \in D^\infty_0(X)$.

**Proof.** By (3.11) the integrand of the integral above is a finite sum of $C^\infty$ densities, each of which is supported on a parametrizable open subset, so we can assume that $\sigma$ itself as this property. Let $V$ be a parametrizable open set containing the support of $\sigma$ and let $\varphi_0 : U \to V$ be a parameterization of $V$. Since $f$ is a diffeomorphism its inverse exists and is a diffeomorphism of $X$ onto $X_1$. Let $V' = f^{-1}(V)$ and $\varphi'_0 = f^{-1} \circ \varphi_0$. Then $\varphi'_0 : U \to V'$ is a parameterization of $V'$. Moreover, $f \circ \varphi'_0 = \varphi$ so if $W_0 = \varphi^{-1}_0(W)$ we have

$$
W_0 = (\varphi_0)^{-1}(f^{-1}(W)) = (\varphi'_0)^{-1}(W')
$$

and by the chain rule we have

$$
\varphi_0^* \sigma = (f \circ \varphi')^* \sigma = (\varphi'_0)^* f^* \sigma
$$

hence

$$
\int_W \sigma = \int_{W_0} \varphi_0^* \sigma = \int_{W_0} (\varphi'_0)^* (f^* \sigma) = \int_{W'} f^* \sigma.
$$

$\square$