Permutohedra, Associahedra, and Beyond

or

Three Formulas for Volumes of Permutohedra

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June 26, 2004

on the occasion of Richard P. Stanley’s Birthday
Permutohedron

\[ P_n(x_1, \ldots, x_{n+1}) := \text{ConvexHull}((x_{w(1)}, \ldots, x_{w(n+1)}) \mid w \in S_{n+1}) \]

This is a convex \( n \)-dimensional polytope in \( H \subset \mathbb{R}^{n+1} \).

**Example:** \( n = 2 \) (type \( A_2 \))

\[ P_2(x_1, x_2, x_3) = \]

More generally, for a Weyl group \( W \), \( P_W(x) := \text{ConvexHull}(w(x) \mid w \in W) \).
**Question:** What is the volume $V_n := \text{Vol } P_n$?

Volume form is normalized so that the volume of a parallelepiped formed by generators of the lattice $\mathbb{Z}^{n+1} \cap H$ is 1.

**Question:** What is the number of lattice points $N_n := P_n \cap \mathbb{Z}^{n+1}$?

We will see that $V_n$ and $N_n$ are polynomials in $x_1, \ldots, x_{n+1}$ of degree $n$. The polynomial $V_n$ is the top homogeneous part of $N_n$. The Ehrhart polynomial of $P_n$ is $E(t) = N_n(tx_1, \ldots, tx_n)$, and $V_n$ is its top coefficient.

We will give 3 totally different formulas for these polynomials.
Special Case:

$$P_n(n + 1, n, \ldots, 1) = \text{ConvexHull}((w(1), ..., w(n + 1)) \mid w \in S_{n+1})$$

is the most symmetric permutohedron.

It is a zonotope = Minkowsky sum of line intervals.

Well-known facts:

- $V_n(n + 1, \ldots, 1) = (n + 1)^{n-1}$ is the number of trees on $n + 1$ labelled vertices. $P_n(n + 1, \ldots, 1)$ can be subdivided into parallelepipeds of unit volume associated with trees. This works for any zonotope.

- $N_n(n + 1, \ldots, 1)$ is the number of forests on $n + 1$ labelled vertices.
First Formula

Fix any distinct numbers $\lambda_1, \ldots, \lambda_{n+1}$ such that $\lambda_1 + \cdots + \lambda_{n+1} = 0$.

\[
V_n(x_1, \ldots, x_{n+1}) = \frac{1}{n!} \sum_{w \in S_{n+1}} \frac{(\lambda_{w(1)} x_1 + \cdots + \lambda_{w(n+1)} x_{n+1})^n}{(\lambda_{w(1)} - \lambda_{w(2)})(\lambda_{w(2)} - \lambda_{w(3)}) \cdots (\lambda_{w(n)} - \lambda_{w(n+1)})}
\]

Notice that the symmetrization in RHS does not depend on $\lambda_i$'s.

Idea of Proof  Use Khovansky-Puchlikov’s method:

➤ Instead of just counting the number of lattice points in $P$, define $[P] = \text{sum of formal exponents } e^a$ over lattice points $a \in P \cap \mathbb{Z}^n$.

➤ Now we can work with unbounded polytopes. For example, for a simplicial cone $C$, the sum $[C]$ is given by a simple rational expression.

➤ Any polytope $P$ can be explicitly presented as an alternating sum of simplicial cones: $[P] = [C_1] \pm [C_2] \pm \cdots$.

Applying this procedure to the permutohedron, we obtain ...
Let \( \alpha_1, \ldots, \alpha_n \) be a system of simple roots for Weyl group \( W \), and let \( L \) be the root lattice.

**Theorem:** For a dominant weight \( \mu \),

\[
[P_W(\mu)] := \sum_{a \in P_W(\mu) \cap (L + \mu)} e^a = \sum_{w \in W} \frac{e^{w(\mu)}}{(1 - e^{-w(\alpha_1)}) \cdots (1 - e^{-w(\alpha_n)})}
\]

Compare this with Weyl’s character formula!

**Note:** LHS is obtained from the character \( ch V_\mu \) of the irrep \( V_\mu \) by replacing all nonzero coefficients with 1. In type \( A \), \( ch V_\mu = \text{Schur polynomial } s_\mu \).

From this expression, one can deduce the First Formula and also its generalizations to other Weyl groups.
Let us use the coordinates \( y_1, \ldots, y_{n+1} \) related \( x_1, \ldots, x_{n+1} \) by

\[
\begin{align*}
y_1 &= -x_1 \\
y_2 &= -x_2 + x_1 \\
y_3 &= -x_3 + 2x_2 - x_1 \\
& \quad \vdots \\
y_{n+1} &= -\binom{n}{0} x_n + \binom{n}{1} x_{n-1} - \cdots \pm \binom{n}{n} x_1
\end{align*}
\]

and write \( V_n = \text{Vol } P_n(x_1, \ldots, x_{n+1}) \) as a polynomial in \( y_1, \ldots, y_{n+1} \).

\textbf{Examples:}

\[
\begin{align*}
V_1 &= \text{Vol } \left( [(x_1, x_2), (x_2, x_1)] \right) = x_1 - x_2 = y_2 \\
V_2 &= \cdots = 3 y_2^2 + 3 y_2 y_3 + \frac{1}{2} y_3^2
\end{align*}
\]
Theorem:

\[
V_n(x_1, \ldots, x_{n+1}) = \frac{1}{n!} \sum_{(S_1, \ldots, S_n)} y_{|S_1|} \cdots y_{|S_n|},
\]

where the sum is over ordered collections of subsets \(S_1, \ldots, S_n \subset [n + 1]\) such that either of the following equivalent conditions is satisfied:

1. For any distinct \(i_1, \ldots, i_k\), we have \(|S_{i_1} \cup \cdots \cup S_{i_k}| \geq k + 1\)
   (cf. Hall’s Marriage Theorem)

2. For any \(j \in [n + 1]\), there is a system of distinct representatives in \(S_1, \ldots, S_n\) that avoids \(j\).

Thus \(n! V_n\) is a polynomial in \(y_2, \ldots, y_{n+1}\) with positive integer coefficients.
This formula can be extended to generalized permutohedra.

Generalized permutohedra are obtained from usual permutohedra by moving faces while preserving all angles.

This is also a generalized permutohedron.
Generalized Permutohedra

Coordinate simplices in \( \mathbb{R}^{n+1} \): \( \Delta_I = \text{ConvexHull}(e_i \mid i \in I) \), for \( I \subseteq [n+1] \).

Let \( Y = \{Y_I\} \) be the collection of variables \( Y_I \geq 0 \) associated with all subsets \( I \subset [n + 1] \). Define

\[
P_n(Y) := \sum_{I \subseteq [n+1]} Y_I \cdot \Delta_I \quad \text{(Minkowsky sum)}
\]

Its combinatorial type depends only on the set of \( I \)'s for which \( Y_I \neq 0 \).

Examples:

- If \( Y_I = y_{|I|} \), then \( P_n(Y) \) is a usual permutohedron.
- If \( Y_I \neq 0 \) iff \( I \) is a consecutive interval, then \( P_n(Y) \) is an associahedron.
- If \( Y_I \neq 0 \) iff \( I \) is a cyclic interval, then \( P_n(Y) \) is a cyclohedron.
- If \( Y_I \neq 0 \) iff \( I \) is a connected set in Dynkin diagram, then \( P_n(Y) \) is a generalized associahedron related to DeConcini-Procesi’s work. (Do not confuse with Fomin-Zelevinsky’s generalized associahedra!)
- If \( Y_I \neq 0 \) iff \( I \) is an initial interval \( \{1, \ldots, i\} \), then \( P_n(Y) \) is the Stanley-Pitman polytope.
**Theorem:** The volume of the generalized permutohedron is given by

\[
\text{Vol} \ P_n(Y) = \frac{1}{n!} \sum_{(S_1, \ldots, S_n)} Y_{S_1} \cdots Y_{S_n},
\]

where \(S_1, \ldots, S_n\) satisfy the same condition.

**Theorem:** The # of lattice points in the generalized permutohedron is

\[
P_n(Y) \cap \mathbb{Z}^{n+1} = \frac{1}{n!} \sum_{(S_1, \ldots, S_n)} \{Y_{S_1} \cdots Y_{S_n}\},
\]

\[
\left\{ \prod_I Y_I^{a_I} \right\} := (Y_{[n+1]}+1)^{\{a_{[n+1]}\}} \prod_{I \neq [n+1]} Y_I^{\{a_I\}}, \quad \text{where } Y^\{a\} = Y(Y+1) \cdots (Y+a-1).
\]

This extends a formula from [Stanley-Pitman] for the volume of their polytope. In this case, the above summation is over parking functions.
We also have a combinatorial description of face structure of generalized permutohedra in terms of nested collections of subsets in $[n + 1]$. This is related to DeConcini-Procesi’s wonderful arrangements.

Not enough time for this now.

The most interesting part of the talk is ...
Let use the coordinates $z_1, \ldots, z_n$ related to $x_1, \ldots, x_{n+1}$ by

$$z_1 = x_1 - x_2, \quad z_2 = x_2 - x_3, \quad \cdots, \quad z_n = x_n - x_{n+1}$$

These coordinates are canonically defined for an arbitrary Weyl group. Then the permutohedron $P_n$ is the Minkowsky sum

$$P_n = z_1 \Delta_{1n} + z_2 \Delta_{2n} + \cdots + z_n \Delta_{nn}$$

of hypersimplices $\Delta_{kn} = P_n(1, \ldots, 1, 0, \ldots, 0)$ (with $k$ 1’s).
This implies

$$\text{Vol } P_n = \sum_{c_1, \ldots, c_n} A_{c_1, \ldots, c_n} \frac{z_1^{c_1}}{c_1!} \cdots \frac{z_n^{c_n}}{c_n!},$$

where the sum is over $c_1, \ldots, c_n \geq 0$, $c_1 + \cdots + c_n = n$, and

$$A_{c_1, \ldots, c_n} = \text{MixedVolume}(\Delta_{1n}^{c_1}, \ldots, \Delta_{nn}^{c_n}) \in \mathbb{Z}_{>0}$$

In particular, $n! V_n$ is a positive integer polynomial in $z_1, \ldots, z_n$. Let us call the integers $A_{c_1, \ldots, c_n}$ the Mixed Eulerian numbers.

**Examples:**

$$V_1 = 1 \ z_1$$
$$V_2 = 1 \ \frac{z_1^2}{2} + 2 \ z_1 z_2 + 1 \ \frac{z_2^2}{2}$$
$$V_3 = 1 \ \frac{z_1^3}{3!} + 2 \ \frac{z_1^2 z_2}{2} + 4 \ z_1 \ \frac{z_2^2}{2} + 4 \ \frac{z_2^3}{3!} + 3 \ \frac{z_1^2 z_3}{2} + 6 \ z_1 z_2 z_3 +$$
$$+ 4 \ \frac{z_2^2 z_3}{2} + 3 \ z_1 \ \frac{z_3^2}{2} + 2 \ z_2 z_3^2 + 1 \ \frac{z_3^3}{3!}$$

(The mixed Eulerian numbers are marked in red.)
Properties of Mixed Eulerian numbers:

- \( A_{c_1,\ldots,c_n} \) are positive integers defined for \( c_1, \ldots, c_n \geq 0, \ c_1 + \cdots + c_n = n \).

- \[
\sum \frac{1}{c_1!\cdots c_n!} A_{c_1,\ldots,c_n} = (n + 1)^{n-1}.
\]

- \( A_{0,\ldots,0,n,0,\ldots,0} \) (\( n \) is in \( k \)-th position) is the usual Eulerian number \( A_{kn} = \# \) permutations in \( S_n \) with \( k \) descents = \( n! \text{Vol} \Delta_{kn} \).

- \( A_{1,\ldots,1} = n! \)

- \( A_{k,0,\ldots,0,n-k} = \binom{n}{k} \)

- \( A_{c_1,\ldots,c_n} = 1^{c_1}2^{c_2}\cdots n^{c_n} \) if \( c_1 + \cdots + c_i \geq i \), for \( i = 1, \ldots, n \).
  There are exactly \( C_n = \frac{1}{n+1} \binom{2n}{n} \) such sequences \((c_1, \ldots, c_n)\).

When I showed these numbers to Richard Stanley, he conjectured that

- \[ \sum A_{c_1,\ldots,c_n} = n! \ C_n. \]

Moreover, he conjectured that \ldots
One can subdivide all sequences \((c_1, \ldots, c_n)\) into \(C_n\) classes such that the sum of mixed Eulerian numbers for each class is \(n!\). For example, \(A_{1,\ldots,1} = n!\) and \(A_{n,0,\ldots,0} + A_{0,n,0,\ldots,0} + A_{0,0,n,\ldots,0} + \cdots + A_{0,\ldots,0,n} = n!\), because the sum of Eulerian numbers \(\sum_k A_{kn}\) is \(n!\).

Let us write \((c_1, \ldots, c_n) \sim (c'_1, \ldots, c'_n)\) iff \((c_1, \ldots, c_n, 0)\) is a cyclic shift of \((c'_1, \ldots, c'_n, 0)\). Stanley conjectured that, for fixed \((c_1, \ldots, c_n)\), we have

\[
\sum_{(c'_1, \ldots, c'_n) \sim (c_1, \ldots, c_n)} A_{c'_1, \ldots, c'_n} = n!
\]

**Exercise:** Check that there are exactly \(C_n\) equivalence classes of sequences.

*Every equivalence class contains exactly one sequence \((c_1, \ldots, c_n)\) such that \(c_1 + \cdots + c_i \geq i\), for \(i = 1, \ldots, n\). (For this sequence, \(A_{c_1,\ldots,c_n} = 1^{c_1} \cdots n^{c_n}\).)*

These conjectures follow from \ldots
**Theorem:** Let \( U_n(z_1, \ldots, z_{n+1}) = \text{Vol} \, P_n \). (It does not depend on \( z_{n+1} \).)

\[
U_n(z_1, \ldots, z_{n+1}) + U_n(z_{n+1}, z_1, \ldots, z_n) + \cdots + U_n(z_2, \ldots, z_{n+1}, z_1) = \\
= (z_1 + \cdots + z_{n+1})^n
\]

This theorem has a simple geometric proof. It extends to any Weyl group. Cyclic shifts come from symmetries of type \( A \) extended Dynkin diagram.

**Idea of Proof:**

The area of blue triangle is \( \frac{1}{6} \) sum of the areas of three hexagons.
Corollary: Fix $z_1, \ldots, z_{n+1}, \lambda_1, \ldots, \lambda_{n+1}$ such that $\lambda_1 + \cdots + \lambda_{n+1} = 0$. Symmetrizing the expression

$$\frac{1}{n!} \frac{(\lambda_1 z_1 + (\lambda_1 + \lambda_2)z_2 + \cdots (\lambda_1 + \cdots + \lambda_{n+1})z_{n+1})^n}{(\lambda_1 - \lambda_2) \cdots (\lambda_n - \lambda_{n+1})}$$

with respect to $(n + 1)!$ permutations of $\lambda_1, \ldots, \lambda_{n+1}$ and $(n + 1)$ cyclic permutations of $z_1, \ldots z_{n+1}$, we obtain

$$(z_1 + \cdots + z_{n+1})^n.$$ 

Problem: Find a direct proof.
Combinatorial interpretation for $A_{c_1,\ldots,c_n}$

A plane binary tree on $n$ nodes

$$T = z_1 z_2 z_3 z_4 z_5 z_6 z_7 z_8$$

(the order is given by green labels)

- The nodes are labelled by $1,\ldots,n$ such that, for a node labelled $l$, labels of all in the left (right) branch are less (greater) than $l$. The labels of all descendants of a node form a consecutive interval $I = [a,b]$.
- We have an increasing labelling of the nodes by $1,\ldots,n$.
- Each node is labeled by $z_i$ such that $i \in I$; $z^T :=$ product of all $z_i$’s.
- The weight of a node labelled by $l$ and $z_i$ with interval $[a,b]$ is $\frac{i-a+1}{l-a+1}$ if $i \leq l$, and $\frac{b-i+1}{b-l+1}$ if $i \geq l$. The weight $wt(T)$ of tree is the product of weights of its nodes.
**Theorem:** The volume of the permutohedron is

\[ V_n = \sum_{T} \text{wt}(T) \cdot z^T \]

where the sum is over plane binary trees with blue, red, and green labels.

Combinatorial interpretation for the mixed Eulerian numbers:

**Theorem:** Let \( z_{i_1} \cdots z_{i_n} = z_1^{c_1} \cdots z_n^{c_n} \). Then

\[ A_{c_1, \ldots, c_n} = \sum_{T} n! \text{wt}(T) \]

over same kind of trees \( T \) such that \( z^T = z_{i_1} \cdots z_{i_n} \) (in this order).

Note that all terms \( n! \text{wt}(T) \) in this formula are positive integer.

Comparing different formulas for \( V_n \), we obtain a lot of interesting combinatorial identities. For example ...
Corollary:

\[(n + 1)^{n-1} = \sum_T \frac{n!}{2^n} \prod_{v \in T} \left(1 + \frac{1}{h(v)}\right),\]

where the sum is over unlabeled plane binary trees \(T\) on \(n\) nodes, and \(h(v)\) denotes the “hook-length” of a node \(v\) in \(T\), i.e., the number of descendants of \(v\) (including \(v\)).

**Example: \(n = 3\)**

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
1 & \quad 2 & \quad 3 \\
\end{align*}
\]

hook-lengths of binary trees

The identity says that

\[(3 + 1)^2 = 3 + 3 + 3 + 3 + 4.\]

**Problem:** Prove this identity combinatorially.