As A. Borel [Ann. of Math. (2) 57 (1953), 115–207; MR 14, 490e] showed, the cohomology ring of the manifold $\text{Fl}_n$ of complete flags in $\mathbb{C}^n$ is isomorphic to the quotient of the polynomial ring in $n$ indeterminates by the ideal of symmetric polynomials. For purposes of geometry, one wants to identify the elements of this quotient ring corresponding to the Schubert classes; this was done by, among others, A. Lascoux and M.-P. Schützenberger [C. R. Acad. Sci. Paris Ser. I Math. 294 (1982), no. 13, 447–450; MR 83e:14039], who identified certain combinatorial representatives called the Schubert polynomials. The (small) quantum cohomology ring of $\text{Fl}_n$ has a similar description: it is the quotient of a polynomial ring in two sets of indeterminates $\mathbb{Z}[q_1, \ldots, q_{n-1}, x_1, \ldots, x_n]$ by an ideal generated by quantum analogues of the elementary symmetric polynomials [A. B. Givental and B. Kim, Comm. Math. Phys. 168 (1995), no. 3, 609–641; MR 96c:58027; I. Ciocan-Fontanine, Internat. Math. Res. Notices 1995, no. 6, 263–277; MR 96h:14071]. Again one would like to identify the elements corresponding to the Schubert classes; equivalently, as the authors explain, one wants to express a given class in the ordinary cohomology ring in terms of the natural generators $x_i$ using the quantum operation. They give an elegant combinatorial solution to this problem, showing that with respect to a carefully chosen basis, the quantum versions of the Schubert polynomials are obtained simply by replacing each elementary symmetric polynomial by its quantum analogue.

Their proof is formal, relying on four properties of the quantum Schubert polynomials. For example, since the structure constants of the quantum product (with respect to the basis of quantum Schubert polynomials) have enumerative significance, they must be nonnegative. They also use several combinatorial properties of their proposed candidates; to establish the crucial “orthogonality” property requires an intricate argument. Given the nature of the proof, they are led to speculate on possible axiomatic characterizations of the quantum Schubert polynomials. They also discuss Grobner basis techniques, and apply them to calculate all Gromov-Witten invariants (the aforementioned structure constants) for $\text{Fl}_3$ and $\text{Fl}_4$.

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