Oscillating Tableaux, $S_p \times S_q$-modules, and Robinson-Schensted-Knuth correspondence

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January 15, 1994

1 Introduction

In the recent time in the works of different authors [4, 5, 6, 7, 11, 14, 17] arose a new interest to the classical Robinson-Schensted-Knuth correspondence [9].

The Robinson-Schensted-Knuth correspondence (RSK) is a bijection between pairs (P, Q) of semi-standard Young tableaux and matrices M with nonnegative integer entries such that the column sums of M give weight of P and the row sums of M give weight of Q (see Corollary 4.5). This correspondence is important in representation theory of the general linear group $GL(N)$ and the symmetric group S_n and in the theory of symmetric functions.

We can view a pair of tableaux (P,Q) of the same shape as a sequence of Young digrams $\alpha_{(0)} = \hat{0} \subset \alpha_{(1)} \subset \ldots \subset \alpha_{(p)} \supset \alpha_{(p+1)} \supset \ldots \supset \alpha_{(k)} = \hat{0}$. In general, consider a sequence of diagrams $\alpha = (\alpha_{(0)}, \alpha_{(1)}, \ldots, \alpha_{(k)})$ such that for all i either $\alpha_{(i)}/\alpha_{(i+1)}$ or $\alpha_{(i+1)}/\alpha_{(i)}$ is a horizontal stripe. Such objects generilize semi-standard Young tableaux (and pairs (P,Q)) and they are called *oscillating tableaux*.

In this paper we use the following notation: $\mathbb{N} := \{0,1,2,\ldots\}$; $s(\beta) := \beta_1 + \beta_2 + \ldots + \beta_k$ for $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$.

We are grateful to S. Fomin, and ??? for useful discussions.

2 Diagrams and tableaux

Recall basic definitions from combinatorics of Young diagrams (see [10]).

A *partition* λ of n is a sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_l = n$. We will also write $\lambda \vdash n$. Let $\mathcal{P}$ denote the set of all partitions. By $\hat{0}$ denote a unique partition of zero.

With each partitions λ we can associate its *Young diagram* which is the set of pairs $(i,j) \in \mathbb{N}^2$ such that $1 \leq j \leq \lambda_i$, $i = 1, 2, \ldots, l$. Pairs (i,j) are arranged on the plane $\mathbb{R}^2$ with i increasing downwards and j increasing from left to right. Young diagrams will be presented in the form of sets of 1×1-boxes centered at (i,j). We denote partitions and the associated Young diagrams by the same letter λ.

Let "$\supset$" be the partitial order on $\mathcal{P}$ by inclusion of Young diagrams, i.e., $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all i. For $\lambda \supset \mu$, *skew Young diagram* λ/μ is the set theoretical difference of the Young diagrams λ and μ. For example, if $\lambda = (6,4,4,1)$, $\mu = (4,3,2)$ then the skew Young diagram λ/μ is the shaded region in Figure 1.

A partition $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ is *conjugate* to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ if their Young diagrams are symmetric to each other with respect to the principal diagonal.

A *horizontal* (respectively, *vertical*) *m-stripe* is a skew Young diagram λ/μ such that every column (respectively, row) contains at most one box of λ/μ and $|\lambda| - |\mu| = m$.

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{N}^k$. A *Young tableau* of *shape* λ/μ and *weight*

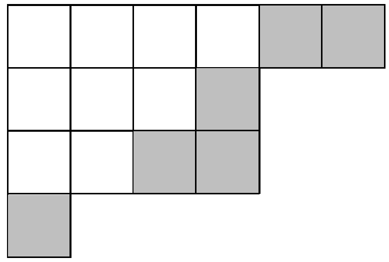

Figure 1: A skew Young diagram λ/μ

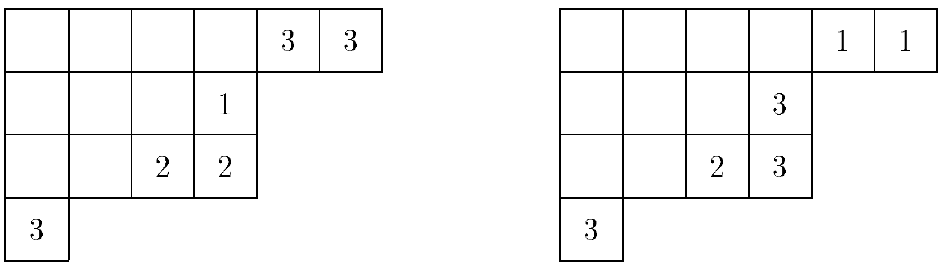

Figure 2: A tableau T and a supertableau S

β is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that $\alpha_{(i-1)} \supset \alpha_(i)$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-stripe for all $i = 1, 2, \ldots, k$. Let $YT(\lambda/\mu, \beta)$ denote the set of all Young tableaux of shape λ/μ and weight β. Note that such tableaux are also called *column-strict* or *semi-standard*. A Young tableau is said to be *standard* if it has weight $\beta = (1, 1, \ldots, 1)$.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{1, -1\}^k$ and β^ε denote the sequence $b = (b_1, b_2, \ldots, b_k)$ in the alphabet $\{m, \overline{m} \mid m \in \mathbb{Z}\}$ such that $b_i = \beta_i$ (respectively $b_i = \overline{\beta}_i$) if $\varepsilon_i = 1$ (respectively $\varepsilon_i = -1$).

A *supertableau* (see [2]) of *shape* λ/μ and *superweight* β^ε is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that $\alpha_{(i-1)} \supset \alpha_{(i)}$ and if $\varepsilon_i = 1$ (respectively, $\varepsilon_i = -1$) then $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal (respectively, vertical) β_i-stripe for all $i = 1, 2, \ldots, k$. Let $ST(\lambda/\mu, b)$ denote the set of all supertableaux of shape λ/μ and superweight $b = \beta^\varepsilon$. It is clear that $ST(\lambda/\mu, \beta^{(1,1,\ldots,1)}) = YT(\lambda/\mu, \beta)$.

When we present tableaux and supertableaux, we insert the integers $k - i + 1$ into the boxes of $\alpha_{(i-1)}/\alpha_{(i)}$ for $i = 1, 2, \ldots, k$. Figure 2 shows examples of a tableau $T \in YT(\lambda/\mu, (1, 2, 3))$ and a supertableau $S \in ST(\lambda/\mu, (2, 1, \overline{3}))$.

3 Oscillating tableaux

We can view tableaux as paths in certain graph $\mathcal{Y}$. The vertices of $\mathcal{Y}$ are Young diagrams and diagrams λ and μ are connected by an edge in $\mathcal{Y}$ if λ/μ (or μ/λ) is a horizontal stripe. Let $\mathcal{Y}_n$ denote the nth *level* of $\mathcal{Y}$, i.e., $\mathcal{Y}_n$ is the set of all diagrams λ with $|\lambda| = n$. We call $\mathcal{Y}$ the *extended Young graph* because it is obtained from the Young graph by adding some edges connecting non-adjacent levels.

It is clear that Young tableaux correspond to decreasing paths in the graph $\mathcal{Y}$. An oscillating tableau is an arbitrary path in $\mathcal{Y}$.

Definition 3.1 *Let λ, μ be partitions and $\beta = (\beta_1, \beta_2, \ldots, \beta_k) \in \mathbb{Z}^k$. An oscillating tableau of shape (λ, μ) and weight β is a sequence of partitions $\alpha = (\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that for all $i = 1, 2, \ldots, k$ the following conditions hold:*

1. *If $\beta_i \geq 0$ then $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-stripe;*

2. *If $\beta_i < 0$ then $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a horizontal $(-\beta_i)$-stripe.*

By $OT(\lambda, \mu, \beta)$ denote the set of all oscillating tableaux of shape (λ, μ) and weight β.

It is clear that $OT(\lambda, \mu, \beta)$ is nonempty only when $|\lambda| - s(\beta) = |\mu|$. If all β_i are nonnegative then $OT(\lambda, \mu, \beta) = YT(\lambda/\mu, \beta)$.

4 Intransitive graphs

Definition 4.1 *Let $\delta = (\delta_1, \delta_2, \ldots, \delta_k) \in \mathbb{Z}^k$ be a sequence such that $s(\delta) = 0$. An intransitive graph of type δ is an oriented graph γ on the vertices $\{1, 2, \ldots, k\}$ (multiple edges allowed) such that:*

1. *If (i, j) is an edge of γ then $i < j$.*

2. *If $\delta_i \geq 0$ then indegree of i is δ_i and outdegree of i is 0.*

3. *If $\delta_i \leq 0$ then outdegree of i is $-\delta_i$ and indegree of i is 0.*

Denote by $G(\delta)$ the set of all intransitive graphs of type δ.

Figure 3: An intransitive graph $\gamma \in G(-2, 1, -2, 0, -2, 2, 3)$

Note that $G(\delta)$ is nonempty if and only if $\sum_{j=1}^{l} \delta_i \leq 0$ for $l = 1, 2, \dots, k$. Figure 3 shows an example of an intransitive graph.

Remark 4.2 *Let $x_1, x_2, \dots, x_k$ be variables. Consider the following q-analogue of Kostant's partition function*

$$P_q = \prod_{i>j} (1 - q e^{x_i - x_j})^{-1} = \sum_{\delta : s(\delta)=0} P_q(\delta) e^{\delta_1 x_1 + \dots + x_k \rho_k}.$$

Then the number $G(\delta)$ of intransitive graphs of type δ is equal to the coefficient of the least power of q in $P_q(\delta)$. So we can view the number $G(\delta)$ as an analogue of $P_q(\delta)$ as $q \to 0$, i.e., "christal analogue of $P_q(\delta)$".

Intransitive graphs are closely related to oscillating tableaux. In Sections 5 and 7 we present several theorem illustrating this connection. Here we formulate a special case which is especially clear.

Theorem 4.3 *Let $\beta \in \mathbb{Z}^k$ be such that $s(\beta) = 0$. Then the number of oscillating tableaux of shape $(\hat{0}, \hat{0})$ and weight β is equal to the number of intransitive graphs of type β*

$$|OT(\hat{0}, \hat{0}, \beta)| = |G(\beta)|.$$

In Section ?? we construct a bijection $\Phi_{\lambda\mu\beta}$ which in the case $\lambda = \mu = \hat{0}$ is is a bijection between $OT(\hat{0}, \hat{0}, \beta)$ and $G(\beta)$.

We call an oscillating tableau of weight $\beta = (\beta_1, \dots, \beta_k)$ *standard* if $\beta_i = \pm 1$ for all i. Clearly, standard oscillating tableux correspond to paths in the Young graph.

Corollary 4.4 *The number of paths in the Young graph from $\hat{0}$ to $\hat{0}$ of length $2k$ is equal to $(2k-1)!! = (2k-1)(2k-3)\dots 1$.*

Proof — If $\beta_i = \pm 1$ for all i then an intransitive graph of type β is a perfect matching. Therefore, by Theorem 4.3 the number of standard tableux of shape $(\hat{0}, \hat{0})$ with weight of length $2k$ is equal to the number perfect matchings on the set of vertices $\{1, 2, \dots, 2k\}$ which is equal to $(2k-1)!!$. $\square$

In the end of this section we show how oscillation tableaux and intransitive graphs are connected with classical Robinson-Shensted-Knuth correspondence [9].

Let $\beta' = (\beta'_1, \beta'_2, \ldots, \beta'_s) \in \mathbb{N}^s$, $\beta'' = (\beta''_1, \beta''_2, \ldots, \beta''_t) \in \mathbb{N}^t$, and β be the sequence $(-\beta'_s, -\beta'_{s-1}, \ldots, -\beta'_1, \beta''_1, \beta''_2, \ldots, \beta''_t) \in \mathbb{Z}^{s+t}$. It is clear that every oscillating tableau $\alpha \in OT(\hat{0}, \hat{0}, \beta)$ can be presented by a pair (P, Q) of Young tableux of the same shape and with weights β' and β'' respectively. We can associate with an intransitive graph $\gamma \in G(\beta)$ the $s{\times}t$-matrix $A = (a_{ij})$ such that a_{ij} is equal to the multiplicity of the edge $(s{+}1{-}i, s{+}j)$ in γ. We get the following corollary of Theorem 4.3.

Corollary 4.5 *Let $\beta' \in \mathbb{N}^s$ and $\beta'' \in \mathbb{N}^t$. Then the number of pairs (P, Q) of Young tableaux of the same shape and with weights β' and β'' respectively is equal to the number of $s{\times}t$-matrices $A = (a_{ij})$ such that*

1. *$a_{ij} \in \mathbb{N}$ for $i = 1, 2, \ldots, s,\ j = 1, 2, \ldots, t$,*
2. *$\sum_j a_{ij} = \beta'_i$ for $i = 1, 2, \ldots, s$,*
3. *$\sum_i a_{ij} = \beta''_j$ for $j = 1, 2, \ldots, t$.*

In [9] D. E. Knuth generalized the constructions of G. de B. Robinson [12] and C. Schencted [13] and obtained a one-to-one correspondence between such pairs (P, Q) and matrices A. In this special case the bijection $\Phi_{\lambda\mu\beta}$ (see Section ??) coincides with Robinson-Schensted-Knuth correspondence.

5 $S_p{\times}S_q$-module $M(p, \beta, q)$

In this section we consider a permutational representation of $S_p{\times}S_q$ in the linear space generated by intransitive graphs. Multiplicities of irreducible components in this representation are given by the numbers of oscillating tableaux.

Let $p, q \in \mathbb{N}$, $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{Z}^k$ such that $p - s(\beta) = q$, $r = p + k$, and $n = p + k + q$. Let $G(p, \beta, q)$ be the set of intransitive graphs of type $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, where

$$\delta_i = \begin{cases} -1 & \text{for } i = 1, \ldots, p, \\ \beta_{i-p} & \text{for } i = p+1, \ldots, r, \\ 1 & \text{for } i = r+1, \ldots, n. \end{cases}$$

The direct product of two symmetric groups $S_p \times S_q$ acts on the graphs $\gamma \in G(p,\beta,q)$ as follows: the group S_p permutes the first p vertices in γ and the group S_q permutes the last q vertices in γ. More precisely, if $g = (\sigma,\rho) \in S_p \times S_q$, $\gamma \in G(p,\beta,q)$ then (i,j) is an edge of graph $g\cdot\gamma$ if and only if $(g^{-1}(i), g^{-1}(j)$ is an edge of γ, where

$$g(s) = \begin{cases} \sigma(s) & s = 1,\dots,p, \\ s & s = p+1,\dots,r, \\ \rho(s-r)+r & s = r+1,\dots,n. \end{cases}$$

Let $M(p,\beta,q)$ be the linear space over $\mathbb{C}$ with basis $\{v_\gamma\}$, $\gamma \in G(p,\beta,q)$. The action of the group $S_p \times S_q$ on $G(p,\beta,q)$ gives a linear representation $M(p,\beta,q)$ of $S_p \times S_q$.

Example 5.1 *Let $p = q$ and $\beta = \emptyset$ be the empty sequence. Then graphs from $G(p,\emptyset,p)$ can be identified with permutations in S_p. In this case $M(p,\emptyset,p)$ is the regular representaion $Reg(S_p)$ of $S_p \times S_p$. That is $M(p,\emptyset,p)$ is isomorphic to the group algebra $\mathbb{C}[S_p]$ on which one copy of S_p acts by left multiplications and the other copy of S_p acts by right multiplications.*

Example 5.2 *Let $q = 0$ and $\beta_i \geq 0$ for all $i = 1,2,\dots,k$. Then a graph $\gamma \in G(p,\beta,0)$ can be identified with the word $w = w_1 w_2 \dots w_p$ in the alphabet $\{1,2,\dots,k\}$ where $w_i = j$ if $(i, p+j)$ is an edge of γ. Clearly, the word w contains β_1 1's, β_2 2's, etc. The symmetric group S_p acts on such words w by permutation of letters w_i. The representation $M_\beta = M(p,\beta,0)$ is the well-known monomial representation, see* [8],

$$M_\beta = Ind^{S_p}_{S_{\beta_1}\times\dots\times S_{\beta_k}} Id,$$

where Id is the identity representation of $S_{\beta_1}\times\dots\times S_{\beta_k}$.

Now we can give a combinatorial interpretation of multiplicities of irreducible components in $M(p,\beta,q)$ in terms of oscillating tableaux.

Let π_λ be the irreducible S_n-module associated with a partition $\lambda \vdash n$ (see [8, 10]). Every irreducible representation of the group $S_p \times S_q$ is of the form $\pi_\lambda \otimes \pi_\mu$, where $|\lambda| = p$ and $|\mu| = q$.

Theorem 5.3

$$M(p,\beta,q) \simeq \sum |OT(\lambda,\mu,\beta)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the sum is over all partitions $\lambda \vdash p$ and $\mu \vdash q$.

The following two Corollaries present classical identities.

For p, q, β such as in Example 5.1 Theorem 5.3 gives

Corollary 5.4

$$Reg(S_p) = \sum_{\lambda \vdash p} \pi_\lambda \otimes \pi_\lambda.$$

This is a standard fact from representation theory of finite groups.

For p, q, β such as in Example 5.2 Theorem 5.3 gives

Corollary 5.5

$$M_\beta = M(p, \beta, 0) = \sum_{\lambda \vdash p} |YT(\lambda, \beta)| \cdot \pi_\lambda$$

This is the classical *Young rule* for decomposition of monomial representations M_β of symmetric groups (see [18, 8, 10]).

Clearly, Theorem 4.3 is a special case of Theorem 5.3 for $p = q = 0$.

6 Proof of Theorem 5.3

Let $\mathcal{M}$ be the category whose objects $\mathrm{Ob}_{\mathcal{M}}$ are finite groups and morphisms $\mathrm{Mor}_{\mathcal{M}}(G, H)$ (or simply $\mathrm{Mor}(G, H)$) from a group G to a group H are equivalence classes of complex finite dimensional $G\times H$-modules. Let $V \in \mathrm{Mor}(G, H)$ and $W \in \mathrm{Mor}(H, K)$, $G, H, K \in \mathrm{Ob}_{\mathcal{M}}$, then composition $V \circ W$ of morphisms V and W is the following $G\times K$-module

$$V \circ W = V \otimes_{\mathbb{C}[H]} W$$

(the tenzor product over the group algebra $\mathbb{C}[H]$). In other words, the tenzor product $V \otimes_{\mathbb{C}} W$ is a $G\times H\times H\times K$-module. Then $V \circ W$ is the $G\times K$-module of H-invariants in $V \otimes_{\mathbb{C}} W$ (with the diagonal action of H on $V \otimes_{\mathbb{C}} W$). The composition is a bilinear operation with respect to the direct sum of modules.

Let $\widehat{G}$ denote the set of equivalence classes of irreducible representations of G. Then any irreducible $G\times H$-module is of the form $\alpha \otimes \beta^*$, where $\alpha \in \widehat{G}$, $\beta \in \widehat{H}$ and β^* denotes the conjugate to β (which is also irreducible). It is clear that these irredusible modules form a $\mathbb{N}$-basis of $\mathrm{Mor}(G, H)$.

Let $\mathrm{Reg}(G)$ be the regular representation of $G\times G$, i. e. $\mathrm{Reg}(G)$ is the group algebra $\mathbb{C}[G]$ on which one copy of G acts by left multiplications and the other copy of G acts by right multiplications.

Figure 4: Composition of graphs

The following proposition presents two simple facts from representation theory of finite groups:

Proposition 6.1 *1. Let $\alpha \in \widehat{G}, \beta \in \widehat{H}, \gamma \in \widehat{H}, \delta \in \widehat{K}$. Then*

$$(\alpha \otimes \beta^*) \circ (\gamma \otimes \delta^*) = \begin{cases} \alpha \otimes \delta^* & \text{if } \beta = \gamma, \\ 0 & \text{if } \beta \neq \gamma. \end{cases}$$

2. *The regular representation $Reg(G) = \sum_{\alpha \in \widehat{G}} \alpha \otimes \alpha^*$ is the identity morphism in the category $\mathcal{M}$ from G to G.*

Now construct a category $\mathcal{T}$. The objects of $\mathcal{T}$ are nonnegative integers $\mathrm{Ob}_{\mathcal{T}} = \mathbb{N}$ and for $p, q \in \mathrm{Ob}_{\mathcal{T}}$ morphisms $\mathrm{Mor}_{\mathcal{T}}(p,q)$ from p to q are sequences $\beta = (\beta_1, \ldots, \beta_k)$ of integers such that $p - s(\beta) = q$ and $p - \sum_{i=1}^{j} \beta_i \geq 0$ for $j = 1, 2, \ldots, k$. The composition of morphisms $\beta' = (\beta'_1, \ldots, \beta'_k)$ and $\beta'' = (\beta''_1, \ldots, \beta''_l)$ is the sequence $\beta' \circ \beta'' = (\beta'_1, \ldots, \beta'_k, \beta''_1, \ldots, \beta''_l)$.

Consider the following maps from $\mathrm{Ob}_{\mathcal{T}}$ to $\mathrm{Ob}_{\mathcal{M}}$ and from $\mathrm{Mor}_{\mathcal{T}}$ to $\mathrm{Mor}_{\mathcal{M}}$

$$M_{ob}: \quad p \in \mathrm{Ob}_{\mathcal{T}} \to S_p \in \mathrm{Ob}_{\mathcal{M}},$$

$$M_{mor}: \quad \beta \in \mathrm{Mor}_{\mathcal{T}}(p,q) \to M(p,\beta,q) \in \mathrm{Mor}_{\mathcal{M}}(S_p, S_q).$$

Theorem 6.2 *These maps give a functor $\mathcal{M}$ from category $\mathcal{T}$ to category $\mathcal{M}$. In other words, if $\beta' \in Mor_{\mathcal{T}}(p,q)$ and $\beta'' \in Mor_{\mathcal{T}}(q,r)$ then $M(p,\beta',q) \circ M(q,\beta'',r) = M(p, \beta' \circ \beta'', r)$.*

Proof — Define an operation of "composition" for intransitive graphs. Let $\gamma' \in G(p,\beta',q)$, $\gamma'' \in G(q,\beta'',r)$, the sequence β' has k elements, and β'' has l elements. Join the vertex $p{+}k{+}i$ of the graph γ' with the vertex i of graph γ'' for $i = 1, 2, \ldots, q$. Delete all these vertices and renumber the remaining vertices by the numbers 1 through $p{+}k{+}l{+}r$ (all vertices of γ' are less then vertices of γ''). As a result we get the graph $\gamma' \circ \gamma'' \in G(p, \beta' \circ \beta'', r)$. See an example on Figure 4.

Let $\{v_{\gamma'}\}$, $\gamma' \in G(p,\beta',q)$ be the basis of $M(p,\beta',q)$ and $\{v_{\gamma''}\}$, $\gamma'' \in G(q,\beta'',r)$ be the basis of $M(q,\beta'',r)$. Then vectors $v_{\gamma'} \otimes v_{\gamma''}$ form a basis of $M(p,\beta',q) \otimes_{\mathbb{C}} M(q,\beta'',r)$. We must select S_q-invariants in this space. To do this we should symmetrize the space $M(p,\beta',q) \otimes_{\mathbb{C}} M(q,\beta'',r)$ by diagonal action of S_q. Let Sym denote this symmetrization. Then we can identify $\mathrm{Sym}(v_{\gamma'} \otimes v_{\gamma''})$ with $v_{\gamma' \circ \gamma''}$. Hence vectors of the type $v_{\gamma' \circ \gamma''}$ generate the representation $M(p, \beta', q) \circ M(q, \beta'', r)$. On the other hand, it is clear that every element of $G(p, \beta' \circ \beta'', r)$ is of the form $\gamma' \circ \gamma''$ and vice versa.

Therefore, $M(p, \beta', q) \circ M(q, \beta'', r) \simeq M(p, \beta' \circ \beta'', r)$. $\square$

Now we are able to prove Theorem 5.3. We will do it in two steps. First, we prove it in the case when the sequence β consists of one number $\beta = (b)$. Then we prove it for arbitrary β.

1. Let $\beta = (-b)$ and $b \geq 0$ (the case when $b \leq 0$ is dual). Then $q = p + b$ and

$$M(p, (-b), q) = \mathrm{Ind}_{S_p \times S_p \times S_b}^{S_p \times S_{p+b}} \mathrm{Reg}(S_p) \otimes \mathrm{Id}_b,$$

where Id_b is the identity representation of S_b. Now

$$\begin{aligned} M(p, (-b), q) \quad &= \mathrm{Ind}_{S_p \times S_p \times S_b}^{S_p \times S_{p+b}} \sum_{\lambda \vdash p} \pi_\lambda \otimes \pi_\lambda \otimes \mathrm{Id}_b \\ = \sum_{\lambda \vdash p} \pi_\lambda \otimes \mathrm{Ind}_{S_p \times S_b}^{S_{p+b}} \pi_\lambda \quad &=^* \sum_{\lambda \vdash p,\, \mu \vdash q} |OT(\lambda, (-b), \mu)| \cdot \pi_\lambda \otimes \pi_\mu. \end{aligned}$$

The first equality is true by Proposition 6.1(2) and the fact that for the symmetric group we have $\pi_\lambda^* = \pi_\lambda$. The equality $(*)$ uses the Pieri rule:

$$\mathrm{Ind}_{S_p \times S_b}^{S_{p+b}} \pi_\lambda = \sum \pi_\mu,$$

where the sum is other all μ such that μ/λ is a horizontal b-stripe, see [8].

2. Let $\beta = (\beta_1, \ldots, \beta_k)$ be a sequence of integers and $p_i = p - \sum_{j=1}^{i} \beta_j$, $q = p_k$. Then

$$\begin{aligned} M(p, \beta, q) \quad &=^{(1)} M(p_0, (\beta_1), p_1) \circ \ldots \circ M(p_{k-1}, (\beta_k), p_k) \\ &=^{(2)} \left(\sum \pi_{\lambda_{(1)}} \otimes \pi_{\mu_{(1)}} \right) \circ \ldots \circ \left(\sum \pi_{\lambda_{(k)}} \otimes \pi_{\mu_{(k)}} \right) \\ &=^{(3)} \sum_{\lambda \vdash p,\, \mu \vdash q} |OT(\lambda, \mu, \beta)| \cdot \pi_\lambda \otimes \pi_\mu, \end{aligned}$$

where in the second line the direct sums are over $\lambda_{(i)} \vdash p_{i-1}$ and $\mu_{(i)} \vdash p_i$ such that $\lambda_{(i)}/\mu_{(i)}$ is a horizontal β_i-stripe (if $\beta_i \geq 0$) or $\mu_{(i)}/\lambda_{(i)}$ is a horizontal $(-\beta_i)$-stripe (if $\beta_i \leq 0$) for all $i = 1, 2, \ldots, k$.

Equality (1) follows from Theorem 6.2; (2) follows from p. 1; (3) follows from Proposition 6.1(1) and definition of oscillating tableaux. $\square$

7 Combinatorial theorem

In this section we give a combinatorial analogue of Theorem 5.3.

A sequence $\tau = (\tau_1, \tau_2, \ldots, \tau_k) \in \mathbb{Z}^k$ is called *normal* if there exist $0 \leq r \leq l \leq k$ such that $\tau_1, \tau_2, \ldots, \tau_r > 0$; $\tau_{r+1} = \ldots = \tau_l = 0$; $\tau_{l+1}, \ldots, \tau_k < 0$. For a sequence $\beta \in \mathbb{Z}^k$, let $\mathrm{nor}(\beta)$ denote the normal sequence obtained from β by shuffling all positive entries of β into the beginning and all negative entries into the end. For example, $\mathrm{nor}(0, -3, 1, -1, 0, -2, 0, 1, 3) = (1, 1, 3, 0, 0, 0, -3, -1, -2)$.

For $\beta, \delta \in \mathbb{Z}^k$ the expression $\delta \prec \beta$ means that for all $i = 1, 2, \ldots, k$ either $0 \leq \delta_i \leq \beta_i$ or $0 \geq \delta_i \geq \beta_i$.

Now we can state the combinatorial theorem.

Theorem 7.1 *Let $\lambda, \mu \in \mathcal{P}$ be some partitions, $\beta \in \mathbb{Z}^k$. Then*

$$|OT(\lambda, \mu, \beta)| = \sum |G(\delta)| \cdot |OT(\lambda, \mu, nor(\beta - \delta))|,$$

where the sum is over all $\delta \in \mathbb{Z}^k$ such that $s(\delta) = 0$ and $\delta \prec \beta$.

In order to deduce Theorem 7.1 from Theorem 5.3 we need one simple lemma.

Lemma 7.2 *Let $p, q \in \mathbb{N}$, $\beta \in \mathbb{Z}^k$ be such that $p - s(\beta) = q$. Then*

$$M(p, \beta, q) = \sum |G(\delta)| \cdot M(p, nor(\beta - \delta), q),$$

where the direct sum is over all $\delta \in \mathbb{Z}^k$ such that $s(\delta) = 0$ and $\delta \prec \beta$.

Proof — Let $\xi \in G(\delta)$, where $\delta = (\delta_1, \ldots, \delta_k) \in \mathbb{Z}^k$, $s(\delta) = 0$. Let $G(p, \delta, q)_\xi$ be the set of graphs from $G(p, \delta, q)$ whose restriction on the vertices $p+1, p+2, \ldots, p+k$ is the graph ξ. If $G(p, \beta, q)_\xi$ is nonempty then $\delta \prec \beta$.

It is clear that when $\delta \prec \beta$ and $\xi \in G(\delta)$ the submodule in $M(p, \beta, q)$ generated by $\{v_\gamma \mid \gamma \in G(p, \beta, q)_\xi\}$ is equivalent to $M(p, \mathrm{nor}(\beta - \delta), q)$. $\square$

Now Theorem 7.1 immediately follows from Theorem 5.3 and Lemma 7.2.

We will give a combinatorial proof of Theorem 7.1. In Section ?? we will construct a bijection $\Phi_{\lambda\mu\beta}$ which establishes a one-to-one correspondence between the following two sets.

$$\Phi_{\lambda\mu\beta} : OT(\lambda, \mu, \beta) \to \coprod G(\delta) \times OT(\lambda, \mu, \mathrm{nor}(\beta - \delta)).$$

Let $\lambda = \mu = \hat{0}$. Then there is a unique oscillating tableau of shape $(\hat{0}, \hat{0})$ of normal weight. Namely, $(\hat{0}, \hat{0}, \ldots, \hat{0}) \in OT(\hat{0}, \hat{0}, (0, 0, \ldots, 0))$. We have $\delta = \beta$ in Theorem 7.1. Hence Theorem 4.3 is a special case of Theorem 7.1.

8 Superanalogue

In this section we give superanalogues of definitions and theorems from Sections 4–7.

Definition 8.1 *Let λ, μ be partitions, $\beta \in \mathbb{Z}^k$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \{1, -1\}^k$. An oscillating supertableau of shape (λ, μ) and superweight $b = \beta^\varepsilon$ (see Section 2) is a sequence of partitions $(\alpha_{(0)} = \lambda, \alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(k)} = \mu)$ such that for all $i = 1, 2, \ldots, k$ the following conditions hold.*

1. *If $\varepsilon_i = 1$ and $\beta_i \geq 0$ then $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a horizontal β_i-stripe;*
2. *If $\varepsilon_i = 1$ and $\beta_i < 0$ then $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a horizontal $(-\beta_i)$-stripe;*
3. *If $\varepsilon_i = -1$ and $\beta_i \geq 0$ then $\alpha_{(i-1)} \supset \alpha_{(i)}$ and $\alpha_{(i-1)}/\alpha_{(i)}$ is a vertical β_i-stripe;*
4. *If $\varepsilon_i = -1$ and $\beta_i < 0$ then $\alpha_{(i)} \supset \alpha_{(i-1)}$ and $\alpha_{(i)}/\alpha_{(i-1)}$ is a vertical $(-\beta_i)$-stripe.*

The set of all oscillating supertableaux of shape (λ, μ) and superweight $b = \beta^\varepsilon$ is denoted by $OST(\lambda, \mu, b)$.

It is clear that $OST(\lambda, \mu, b)$ is nonempty only when $|\lambda| - s(\beta) = |\mu|$. If all $\beta_i \geq 0$ then $OST(\lambda, \mu, \beta^\varepsilon) = ST(\lambda/\mu, \beta^\varepsilon)$. And $OST(\lambda, \mu, \beta^{(1,1\ldots,1)}) = OT(\lambda, \mu, \beta)$.

Definition 8.2 *Let $\delta \in \mathbb{Z}^k$ be such that $s(\delta) = 0$ and $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k) \in \{1, -1\}^k$. An intransitive graph of supertype $d = \delta^\epsilon$ is an oriented graph γ on the set of vertices $\{1, 2, \ldots, k\}$ satistying the conditions 1–3 of Definition 4.1 and also the condition:*

4. *If $\epsilon_i \neq \epsilon_j$ then γ contains at most one edge (i, j).*

Let $SG(\delta^\epsilon)$ be the set of all such graphs.

The following algebra $\mathcal{A}(\epsilon)$ is closely related to Definition 8.2.

Definition 8.3 *Let $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_k) \in \{1, -1\}^k$. The algebra $\mathcal{A}(\epsilon)$ generated by variables $x_{ij}, 1 \le i < j \le k$ with the following relations.*

1. $x_{ij}\, x_{jr} = 0$ *for any* $1 \le i < j < r \le k$,
2. $x_{ij}\, x_{lm} = (-1)^{\sigma_{ij}\sigma_{lm}} x_{lm}\, x_{ij}$, *where*

$$\sigma_{ij} = \begin{cases} 0 & \epsilon_i = \epsilon_j, \\ 1 & \epsilon_i \ne \epsilon_j. \end{cases}$$

Relation 2 implies that x_{ij} with $\sigma_{ij} = 0$ are commutative variables and x_{lm} with $\sigma_{lm} = 1$ are anticommutative variables.

For any oriented graph γ on the set of vertices $\{1, 2, \dots, k\}$ we can construct (up to a sign) a monomial m_γ in the algebra $\mathcal{A}(\epsilon)$:

$$m_\gamma = \pm \prod x_{ij},$$

where the product is over all edges (i, j) of graph γ.

Nonzero monomials in $\mathcal{A}(\epsilon)$ correspond to intransitive graphs of type β^ϵ with fixed ϵ and arbitrary β. Indeed, condition 4.1(2) corresponds to condition 8.3(1) and 8.2(4) corresponds to the fact that $x_{lm}^2 = 0$ for an anticommutative variable x_{lm} with $\sigma_{lm} = 1$.

Let $\mathcal{A}_\delta(\epsilon)$ denote the subspace of $\mathcal{A}(\epsilon)$ which is generated (as a linear space) by monomials m_γ for $\gamma \in SG(\delta^\epsilon)$. It is clear that $\mathcal{A}(\epsilon) = \bigoplus_\delta \mathcal{A}_\delta(\epsilon)$. Let $p, q \in \mathbb{N}$, $\beta = (\beta_1, \dots, \beta_k)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{1, -1\}^k$, and $\psi, \omega \in \{1, -1\}$. Suppose that

$$\begin{aligned} \delta &= (\underbrace{-1, -1, \dots, -1}_{p \text{ times}}, \beta_1, \beta_2, \dots, \beta_k, \underbrace{1, 1, \dots, 1}_{q \text{ times}}); \\ \epsilon &= (\underbrace{\psi, \psi, \dots, \psi}_{p \text{ times}}, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \underbrace{\omega, \omega, \dots, \omega}_{q \text{ times}}). \end{aligned}$$

Let $SG(\mathrm{p}, \beta^\varepsilon, \mathrm{q})$ be the set of intransitive graphs of supertype δ^ϵ. Denote by $M(\mathrm{p}, \beta^\varepsilon, \mathrm{q})$ the subspace $\mathcal{A}_\delta(\epsilon)$, where $\mathrm{p} = p^\psi$ and $\mathrm{q} = q^\omega$. Then $\{m_\gamma : \gamma \in SG(\mathrm{p}, \beta^\varepsilon, \mathrm{q})\}$ is a basis of the space $M(\mathrm{p}, \beta^\varepsilon, \mathrm{q})$.

The group $S_p \times S_q$ acts on this space, cf. Section 5. The symmetric group S_p permutes the first index of variables x_{ij} with $i = 1, 2, \dots, p$ and S_q permutes the second index of variables x_{ij} with $j = p+k+1, \dots, p+k+q$.

The following example gives an odd analogue of the regular representation of S_p (see Example 5.1).

Example 8.4 *Let $\beta^\varepsilon = \emptyset$ be the empty sequence,* $\mathrm{p} = p$ *and and* $\mathrm{q} = \overline{p}$*, $p \in \mathbb{N}$. Then $M(p, \emptyset, \overline{p})$ is the representation of $S_p \times S_p$ on the group algebra $\mathbb{C}[S_p]$ given by the formula*

$$(\sigma, \pi) \cdot f = sgn(\sigma\pi^{-1})\, \sigma f \pi^{-1},$$

where $(\sigma, \pi) \in S_p \times S_p$, $f \in \mathbb{C}[S_p]$ and sgn denotes the sign of permutation. Denote this representation by Alt_p.

We use the following notation. For a partition $\lambda \in \mathcal{P}$ and $\psi \in \{1, -1\}$, $\lambda^\psi = \lambda$ if $\psi = 1$ and $\lambda^\psi = \lambda'$ (the conjugate partition) if $\psi = -1$.

Now we can present a superanalogue of Theorem 5.3.

Theorem 8.5

$$M(p^\psi, \beta^\varepsilon, q^\omega) \simeq \sum |OST(\lambda^\psi, \mu^\omega, \beta^\varepsilon)| \cdot \pi_\lambda \otimes \pi_\mu,$$

where the sum is over all partitions $\lambda \vdash p$ and $\mu \vdash q$.

For p, q, β^ϵ such as in Example 8.4 we have by Theorem 8.5

Corollary 8.6

$$Alt_p = \sum_{\lambda \vdash p} \pi_\lambda \otimes \pi_{\lambda'}.$$

This is an odd analogue of Corollary 5.4. Of course this formula easily follows from definition of Alt_p.

Sketch of proof of Theorem 8.5 — The proof is analogous to the proof of Theorem 5.3. The only difference is the definition of "composition" for intransitive graphs. If we define the composition as in Section 6 then it may happen that the composition of two graphs $\gamma' \in SG(\mathrm{p}, b', \mathrm{q})$ and $\gamma'' \in SG(\mathrm{q}, b'', \mathrm{r})$ is not a graph from $SG(\mathrm{p}, b' \circ b'', \mathrm{r})$. We define "supercomposition" $\gamma' \circ^{\mathrm{S}} \gamma''$ of graphs γ' and γ'' by

$$\gamma' \circ^{\mathrm{S}} \gamma'' = \begin{cases} \gamma' \circ \gamma'' & \text{if } \gamma' \circ \gamma'' \in SG(\mathrm{p}, b' \circ b'', \mathrm{r}), \\ 0 & \text{otherwise.} \end{cases}$$

This convention is consistent with interpretation of composition in terms of symmetrization. Indeed, if $\gamma' \circ \gamma''$ is not in $SG(\mathrm{p}, b' \circ b'', \mathrm{r})$ then $\mathrm{Sym}(m(\gamma') \otimes m(\gamma'')) = 0$. $\square$

Now we give a superanalogue of Theorem 7.1. Let $b = (b_1, b_2, \ldots, b_k) = \beta^\varepsilon$ (see Section 2). Let $\mathrm{nor}(b)$ denote the word obtained from the word $b = (b_1, b_2, \ldots, b_k)$ by shuffling all negative entries into the beginning and all positive entries into the end. For example, $\mathrm{nor}(0, \overline{3}, -1, \overline{1}, 0, 2, \overline{0}, -\overline{1}, -3) = (-1, -\overline{1}, -3, 0, 0, \overline{0}, \overline{3}, \overline{1}, 2)$.

Theorem 8.7 *Let $\lambda, \mu \in \mathcal{P}$ be some partitions, $\beta \in \mathbb{Z}^k$, $\varepsilon \in \{1, -1\}^k$. Then*

$$|OST(\lambda, \mu, \beta^\varepsilon)| = \sum |SG(\delta^\varepsilon)| \cdot |OST(\lambda, \mu, nor((\beta - \delta)^\varepsilon))|,$$

where the summation is over all $\delta \in \mathbb{Z}^k$ such that $s(\delta) = 0$ and $\delta \prec \beta$.

This theorem can be deduced from Theorem 8.5 in the same way as Theorem 7.1 from Theorem 5.3.

In Section ?? we will construct a bijection

$$\Phi^{super}_{\lambda\mu b} : OST(\lambda, \mu, \beta^\varepsilon) \to \coprod_{\delta \prec \beta} SG(\delta^\varepsilon) \times OST(\lambda, \mu, \mathrm{nor}((\beta - \delta)^\varepsilon)).$$

This will give a combinatorial proof of Theorem 8.5.

If $\lambda = \mu = \hat{0}$ then Theorem 8.7 implies the following

Corollary 8.8 *Let $\beta \in \mathbb{Z}^k$ be such that $s(\beta) = 0$, $\varepsilon \in \{1, -1\}^k$. Then the number of oscillating tableaux of shape $(\hat{0}, \hat{0})$ and superweight $b = \beta^\varepsilon$ is equal to the number of intransitive graphs of sypertype b*

$$|OST(\hat{0}, \hat{0}, b)| = |G(b)|.$$

Let $\beta' \in \mathbb{N}^s$, $\beta'' \in \mathbb{N}^t$, $\beta = (-\beta'_s, -\beta'_{s-1}, \ldots, -\beta'_1, \beta''_1, \beta''_2, \ldots, \beta''_t)$, and $\varepsilon = (-1, -1, \ldots, -1, 1, 1, \ldots, 1)$ (s -1's and t 1's). It is clear that oscillating supertableaux of shape $(\hat{0}, \hat{0})$ and superweight β^ε correspond to pairs (P, Q) of Young tableaux with conjugate shapes and with weights β', β'' respectively, cf. Section 4.

We can identify an intransitive graph $\gamma \in SG(\beta^\varepsilon)$ with a $s \times t$-matrix $A = (a_{ij})$ satisfying conditions 1–3 of Corollary 4.5 and such that $a_{ij} = 0$ or 1 for all i and j. We get the following

Corollary 8.9 *Let $\beta' \in \mathbb{N}^s$ and $\beta'' \in \mathbb{N}^t$. Then the number of pairs of tableaux (P,Q) with conjugated shapes and with weights β' and β'' respectively is equal to the number of $s\times t$-matrices satisfying the conditions 1–3 of Corollary 4.5 with all entries equal to 0 or 1.*

Knuth in [9] construct also an odd analogue of RSK-correspondence which is a bijection between the set of such $s\times t$-matrices and the set of such pairs of tableaux (P,Q). In this special case the bijection $\Phi^{super}_{\lambda\mu b}$ coincides with Knuth's correspondence.

9 Increasing and decreasing operators

First we give another description of the category $\mathcal{M}$ from Section 6.

Let G be a finite group. By $\mathrm{Rep}(G)$ denote the set of equivalence classes of complex finite dimensional representations of G. It is clear that $\mathrm{Rep}(G) = \mathrm{Mor}_{\mathcal{M}}(\{\mathrm{id}\}, G)$ (see Section 6), where $\{\mathrm{id}\}$ denote the group with one element id.

Let $W \in \mathrm{Mor}_{\mathcal{M}}(G,H)$. Consider the $\mathbb{N}$-linear map $\langle W\rangle$ from $\mathrm{Rep}(G)$ to $\mathrm{Rep}(H)$ which is defined by $\langle W\rangle V = V\circ W$, where $V \in \mathrm{Rep}(G) = \mathrm{Mor}_{\mathcal{M}}(\{\mathrm{id}\}, G)$. On the other hand, if we know a map $\langle W\rangle$ then we can reconstruct the morphism W in $\mathcal{M}$.

By R denote the direct sum $R = \mathrm{Rep}(S_0)\oplus\mathrm{Rep}(S_1)\oplus\mathrm{Rep}(S_2)\oplus\ldots$.

Let $\langle M(p,b,q)\rangle$ be the operator from $\mathrm{Rep}(S_p)$ to $\mathrm{Rep}(S_q)$ which corresponds to $S_p\times S_q$-module $M(p,b,q)$. Recall that $b = \beta^\varepsilon$ is a sequence in the alphabet $\{m, \overline{m} \mid m \in \mathbb{Z}\})$. Let $\langle b\rangle$ be the endomorpism of R such that $\langle b\rangle = \sum \langle M(p,b,q)\rangle$, where the sum is over $p - s(\beta) = q$. In the case when the sequence b has only one element m or $\overline{m}$, $m \in \mathbb{Z}$, we denote these operators by $\langle m\rangle$ or $\langle\overline{m}\rangle$. It is clear from Section 8 that $\langle (b_1, b_2, \ldots, b_k)\rangle = \langle b_1\rangle\cdot\langle b_2\rangle\cdot\ldots\cdot\langle b_k\rangle$.

If $n \in \mathbb{N}$ then we call operators $\langle n\rangle$ and $\langle\overline{n}\rangle$ *increasing* and denote them by $I(n)$ or $I(\overline{n})$. If $-n \in \mathbb{N}$ then we call operators $\langle n\rangle$ and $\langle\,\overline{n}\,\rangle$ *decreasing* and denote them $D(n)$ or $D(\overline{n})$. The following description of operators $I(n)$, $I(\overline{n})$, $D(n)$, and $D(\overline{n})$ follows from Sections 6 and 8.

Let $V \in \mathrm{Rep}(S_p)$. Then

$$\begin{aligned} I(n)\cdot V &= \mathrm{Ind}_{S_p}^{S_{p+n}} V; \\ I(\overline{n})\cdot V &= \mathrm{Ind}_{S_p\times S_n}^{Sp+n} (V\otimes \mathrm{sgn}_n), \end{aligned}$$

where sgn_n is the sign representation of S_n.

Let $V \in \mathrm{Rep}(S_{p+n})$. Then

$$
\begin{aligned}
D(n) \cdot V &= \mathrm{Inv}_n(\mathrm{Res}^{S_{p+n}}_{S_p \times S_n} V); \\
D(\overline{n}) \cdot V &= \mathrm{Skew}_n(\mathrm{Res}^{S_{p+n}}_{S_p \times S_n} V),
\end{aligned}
$$

where Inv_n is the space of S_n-invariants and Skew_n is the space of skew invariants of S_n.

The space R has the basis $\{\pi_\lambda \mid \lambda \in \mathcal{P}\}$ consisting of all irreducible representations of all symmetric groups. Therefore a linear operator on the space R can be represented as an infinite matrix indexed by partitions.

All increasing and decreasing operators in coordinates are given below.

$$
\begin{aligned}
I(n)_{\lambda\mu} &= \begin{cases} 1 & \text{if } \lambda \supset \mu \text{ and } \lambda/\mu \text{ is a horizontal } n\text{-stripe}, \\ 0 & \text{otherwise}, \end{cases} \\
D(n)_{\lambda\mu} &= \begin{cases} 1 & \text{if } \mu \supset \lambda \text{ and } \mu/\lambda \text{ is a horizontal } n\text{-stripe}, \\ 0 & \text{otherwise}, \end{cases} \\
I(\overline{n})_{\lambda\mu} &= \begin{cases} 1 & \text{if } \lambda \supset \mu \text{ and } \lambda/\mu \text{ is a vertical } n\text{-stripe}, \\ 0 & \text{otherwise}, \end{cases} \\
D(\overline{n})_{\lambda\mu} &= \begin{cases} 1 & \text{if } \mu \supset \lambda \text{ and } \mu/\lambda \text{ is a vertical } n\text{-stripe}, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
$$

It is clear that $\langle b\rangle_{\lambda\mu} = (\langle b_1\rangle \cdot \langle b_2\rangle \cdot \ldots \cdot \langle b_k\rangle)_{\lambda\mu} = |OST(\lambda, b, \mu)|$.

All increasing operators commute and all decreasing operators commute. But increasing and decreasing operators do not commute with each other. The following theorem gives the relations between these operators. Here $[a, b] = ab - ba$ denotes the commutator of operators.

Theorem 9.1 *Let $m, n \in \mathbb{N}$. The following relations hold.*

1. $[I(m), I(n)] = [I(\overline{m}), I(\overline{n})] = [D(m), D(n)] = [D(\overline{m}), D(\overline{n})] = 0.$
2. $[I(m), I(\overline{n})] = [D(m), D(\overline{n})] = 0.$
3. $[I(m+1), D(n+1)] = I(m)D(n),\ [I(\overline{m+1}), D(\overline{n+1})] = I(\overline{m})D(\overline{n}).$
4. $[I(m+1), D(\overline{n+1})] = D(\overline{n})I(m),\ [I(\overline{m+1}), D(n+1)] = D(n)I(\overline{m}).$

In the following section we give a combinatorial proof of Theorem 9.1.

10 Local bijections

Let $m, n \in \mathbb{N}$. In this section we construct the following four bijections:

1. $\psi_1 : YT(\lambda/\nu, (m, n)) \to YT(\lambda/\nu, (n, m))$,
2. $\psi_2 : ST(\lambda/\mu, (m, \overline{n})) \to ST(\lambda/\nu, (\overline{n}, m))$,
3. $\psi_3 : OT(\lambda, \nu, (-m, n)) \to \coprod_{0 \le k \le \min(m,n)} OT(\lambda, \nu, (n-k, -m+k))$,
4. $\psi_4 : OST(\lambda, \nu, (-m, \overline{n})) \to \coprod_{0 \le k \le \min(1,m,n)} OST(\lambda, \nu, (\overline{n-k}, -m+k))$.

It is clear that these bijections are sufficient to prove Theorem 9.1. Later we will use bijections ψ_3 and ψ_4 in combinatorial proofs of Theorems 7.1 and 8.7

In all examples, when displaying an (oscillating) (super)tableau $\alpha = (\lambda, \mu, \nu)$, we insert 2's into the boxes of the skew diagram λ/μ (or μ/λ) and 1's into the boxes of μ/ν (or ν/μ). The symbol 1/2 in a box means that we insert simultaneously integers 1 and 2 into this box.

We say that a skew diagram λ/μ *falls into a disjoint union* of skew diagrams $\tau_1, \tau_2, \ldots, \tau_l$ if $\lambda/\mu = \cup_i \tau_i$ and for all $1 \le i < j \le l$ any box of τ_j is below and the to the left of any box of τ_j. For example, the skew diagram on Figure 1 falls into a disjoint union of three diagrams. We also say that a (super)tableau of shape λ/μ falls into a disjoint union of so does the shape λ/μ.

Constructions:

1. Let $\alpha = (\lambda, \mu, \nu) \in YT(\lambda/\mu, (m, n))$, $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$, and $\nu = (\nu_1, \nu_2, \ldots)$. Then we have $\lambda_i \ge \mu_i \ge \lambda_{i+1}$, $i = 1, 2 \ldots$; and $\mu_i \ge \nu_i \ge \mu_{i+1}$, $i = 1, 2, \ldots$. Set by convention $\nu_0 = \infty$. On the following diagram arrow $x \to y$ denotes the inequality $x \ge y$.

$$
\begin{array}{cccccccccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \ldots & \lambda_i & & & & \lambda_{i+1} & \ldots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & & & \searrow & & \nearrow & & \\
 & & \mu_1 & & & & \mu_2 & & & \ldots & & & \mu_i & & & \ldots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & & & \nearrow & & \searrow & & \\
\infty & & & & \nu_1 & & & & \nu_2 & \ldots & \nu_{i-1} & & & & \nu_i & \ldots
\end{array}
$$

Let $a_i = \min(\lambda_i, \nu_{i-1})$ and $b_i = \max(\lambda_{i+1}, \nu_i)$, $i = 1, 2, \ldots$. Then $a_i \ge \mu_i \ge b_i$. Set $\widetilde{\mu}_i = a_i + b_i - \mu_i$, $i = 1, 2, \ldots$, i.e., $\widetilde{\mu}_i$ is symmetric to μ_i in the interval (b_i, a_i).

Figure 5: Bijection ψ_1

Figure 6: Bijection ψ_3

Now $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ is a partition and $\widetilde{\alpha} = (\lambda, \widetilde{\mu}, \nu) \in YT(\lambda/\mu, (n, m))$. Define $\psi_1 : \alpha \mapsto \widetilde{\alpha}$. It is easy to see that ψ_1 is a bijection between the sets $YT(\lambda/\mu, (m, n))$ and $YT(\lambda/\mu, (n, m))$. Figure 5 shows an example of the bijection ψ_1.

2. Let $\alpha = (\lambda, \mu, \nu) \in ST(\lambda/\mu, (m, \overline{n}))$ ...

3. Let $\alpha = (\lambda, \mu, \nu) \in OT(\lambda, \mu, (-m, n)$, $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$, and $\nu = (\nu_1, \nu_2, \ldots)$. Then we have $\mu_i \geq \lambda_i \geq \mu_{i+1}$, $\mu_i \geq \nu_i \geq \mu_{i+1}$, $i = 1, 2, \ldots$; $|\mu| - |\lambda| = m$, and $|\mu| - |\nu| = n$.

$$\begin{array}{cccccccccccccccccc}
 & & \lambda_1 & & & & \lambda_2 & & & & \lambda_3 & \cdots & \lambda_i & & & & \lambda_{i+1} & \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & \searrow & & \nearrow & & \\
\mu_1 & & & & \mu_2 & & & & \mu_3 & & & \cdots & & & \mu_{i+1} & & & \cdots \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & & \nearrow & & \searrow & & \\
 & & \nu_1 & & & & \nu_2 & & & & \nu_3 & \cdots & \nu_i & & & & \nu_{i+1} & \cdots
\end{array}$$

Let $a_i = \min(\lambda_i, \nu_i)$ and $b_i = \max(\lambda_{i+1}, \nu_{i+1})$, $i = 1, 2 \ldots$. Then $a_i \geq \mu_{i+1} \geq b_i$. Set $\widetilde{\mu}_i = a_i + b_i - \mu_{i+1}$, $i = 1, 2, \ldots$ (cf. p. 1) and $k = \mu_1 - \min(\lambda_1, \nu_1)$. Clearly, $0 \leq k \leq \min(n, m)$.

Now $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ is a partition and $\widetilde{\alpha} = (\lambda, \widetilde{\mu}, \nu) \in OT(\lambda, \mu, (n - k, -m + k))$. We define $\psi_3 : \alpha \mapsto \widetilde{\alpha}$. Then ψ_3 gives a bijection between the sets $OT(\lambda, \mu, (-m, n))$ and $\coprod_k OT(\lambda, \mu, (n-k, -m+k))$, $0 \leq k \leq \min(m, n)$. Indeed, if we have a partition $\widetilde{\mu} = (\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots)$ and $0 \leq k \leq \min(m, n)$ then we can reconstruct μ setting $\mu_1 = k + \min(\lambda_1, \nu_1)$ and $\mu_{i+1} = a_i + b_i - \widetilde{\mu}_i$, $i = 1, 2, \ldots$. See an example of the bijection ψ_3 on Figure 6.

4. Let $\alpha = (\lambda, \mu, \nu) \in OST(\lambda, \nu, (-m, \overline{n}))$ ...

11 Generalized Gelfand-Tsetlin patterns

Let $\alpha = (\alpha_{(0)}, \alpha_{(1)}, \ldots, \alpha_{(k)}) \in OT(\lambda, \mu, \beta)$ be an oscillating tableau of weight $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$. Let $w = w_1 w_2 \ldots w_k$ be a word in the alphabet $\{+, -\}$

0	0	1	1		
0	1	1	1	1	
1	1	1	2	2	2
	2	2	2	2	3
		3	4		

Figure 7:

such that if β_i is positive (negative) then $w_i = +$ ($w_i = -$), $i = 1, 2, \ldots, k$. Let $\rho(i)$ be the number of $+$'s in the word $w_1 w_2 \ldots w_i$, $i = 1, 2, \ldots, k$.

The *generalized Gelfand-Tsetlin pattern* P of type w corresponding to the oscillating tableau α is the two-dimensional array $P = \{p_{ij}\}$, where $i = 1, 2, \ldots, k$, $j \geq \rho(i)$, and $p_{ij} = \alpha_{(i)j-\rho(i)}$. For example, a generalized Gelfand-Tsetlin pattern of type $w = {+}{+}{-}\ldots$ is an array of the following form (as above $x \to y$ means $x \geq y$).

$$\begin{array}{ccccccccccccccc}
\alpha_{(0)1} & & & & \alpha_{(0)2} & & & & \alpha_{(0)3} & & & & \alpha_{(0)4} & & \cdots \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\
& & \alpha_{(1)1} & & & & \alpha_{(1)2} & & & & \alpha_{(1)3} & & & & \cdots \\
& & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\
& & & & \alpha_{(2)1} & & & & \alpha_{(2)2} & & & & \alpha_{(2)3} & & \cdots \\
& & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\
& & \alpha_{(3)1} & & & & \alpha_{(3)2} & & & & \alpha_{(3)3} & & & & \cdots \\
& & & & & & & & & & & & & & \\
. & & . & & . & & . & & . & & . & & . & & \ddots
\end{array}$$

Note that standard Gelfand-Tsetlin patterns have type $w = {+}{+}{+}\ldots$ in our terminology.

We can present a generalized Gelfand-Tsetlin pattern P (and the corresponding oscillating tableau) in more convinient form as a plane partition with cutted off corners. For example, Figure 7 presents the oscillating tableau

$$((211), (3211), (221), (211), (421), (321)).$$

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