IMRN International Mathematics Research Notices
1999, No. 8

Mixed Bruhat Operators and Yang-Baxter Equations for Weyl Groups

Francesco Brenti, Sergey Fomin, and Alexander Postnikov

1 Introduction

We introduce and study a family of operators which act in the group algebra of a Weyl group W and provide a multiparameter solution to the quantum Yang-Baxter equations of the corresponding type. These operators are then used to derive new combinatorial properties of W and to obtain new proofs of known results concerning the Bruhat order of W.

The paper is organized as follows. Section 2 is devoted to preliminaries on Coxeter groups and associated Yang-Baxter equations.

In Theorem 3.1 of Section 3, we describe our solution of these equations.

In Section 4, we consider a certain limiting case of our solution, which leads to the *quantum Bruhat operators*. These operators play an important role in the explicit description of the (small) quantum cohomology ring of G/B. Section 5 contains the proof of Theorem 3.1.

Section 6 is devoted to combinatorial applications of our operators. For an arbitrary element $u \in W$, we define a graded partial order on W called the *tilted Bruhat order*; this partial order has unique minimal element u. (The usual Bruhat order corresponds to the special case where $u = e$, the identity element.) We then prove that tilted Bruhat orders are lexicographically shellable graded posets whose every interval is Eulerian. This generalizes the well-known results of D.-N. Verma, A. Björner, M. Wachs, and M. Dyer.

Received 20 November 1998. Revision received 16 December 1998.
Communicated by Andrei Zelevinsky.

2 Coxeter groups and Yang-Baxter equations

We first recall some standard terminology and notation related to Coxeter groups. Missing details can be found in [5], [17]. Let (W, S) be a finite *Coxeter system*. Thus W is a finite Coxeter group generated by the set S of its *simple reflections*, which satisfy the relations $(st)^{m(s,t)} = 1$, for $s, t \in S$. The parameters $m(s,t)$ are nonnegative integers such that $m(s,s) = 1$ and $m(s,t) = m(t,s) > 1$ for $s \neq t$.

For an element $w \in W$, an expansion $w = s_1 \cdots s_l$ of minimal possible length l is called a *reduced decomposition*. The number $l = \ell(w)$ is the *length* of w. The elements of the set $T = \{wsw^{-1}\colon\ w \in W,\ s \in S\}$ are the *reflections* of W.

The *Bruhat order* on W is defined as follows: $u \leq v$ if and only if there exist $t_1, \dots, t_r \in T$ such that $t_r \dots t_1\, u = v$ and $\ell(t_i \dots t_1\, u) > \ell(t_{i-1} \dots t_1\, u)$ for $i = 1, \dots, r$.

A subgroup W' of W generated by a subset of T is called a *reflection subgroup*. The group W' is again a Coxeter group, with a distinguished set $S' \subset W$ of *canonical generators*; see [10], [11], or [17, Section 8.2]. We are only interested in the case where W' is a *dihedral reflection subgroup*; i.e., S' has two elements. A dihedral reflection subgroup is *maximal* if it is not contained in another such subgroup.

In this paper, we mostly deal with the case where W is a *Weyl group*, with the associated *root system* Φ in a Euclidean space V. The group W acts in V by orthogonal transformations. Each root $\alpha \in \Phi$ corresponds to an element $s_\alpha \in T$, which acts as a reflection with respect to the hyperplane orthogonal to α. Simple roots correspond to the Coxeter generators in S. Dihedral reflection subgroups $W' \subset W$ are in one-to-one correspondence with rank 2 root subsystems $\Phi' \subset \Phi$. Such a subgroup is maximal if Φ' is obtained by intersecting Φ with a two-dimensional plane. Canonical generators of W' correspond to the simple roots of Φ', i.e., to indecomposable positive roots in this subsystem.

Let W' be a dihedral reflection subgroup of order $2k$. Its canonical generators a and b satisfy

$$\underbrace{ababa\cdots}_{k} = \underbrace{babab\cdots}_{k}.$$

Thus the set of reflections $T' = T \cap W'$ consists of k elements $a, aba, ababa, \dots, babab$, bab, b.

Let w_o be the longest element in W, and denote $\ell(w_o) = |T|$ by N. Following M. Dyer [13], we say that a bijection $\varphi\colon\ T \to \{1, \dots, N\}$ is a *reflection ordering* if, for any dihedral reflection subgroup W' with canonical generators a and b, the sequence

$$\varphi(a), \varphi(aba), \varphi(ababa), \dots, \varphi(babab), \varphi(bab), \varphi(b)$$

is either increasing or decreasing. (It is enough to require this for every maximal dihedral subgroup.) Reflection orderings are in bijective correspondence with reduced decompositions of w_o, namely, φ is a reflection ordering if and only if there exists a reduced decomposition $w_o = s_1 \dots s_N$ such that

$$\varphi^{-1}(j) = s_N s_{N-1} \dots s_{j+1} s_j s_{j+1} \dots s_{N-1} s_N \tag{2.1}$$

for $j = 1, \dots N$. (See [13, Proposition 2.13].)

Definition 2.1. A family $\{R_\tau\}_{\tau \in T}$ of elements of a monoid is called an (extensible) *solution to the Yang-Baxter equations* for W if, for any dihedral reflection subgroup W' of W with canonical generators a and b, we have

$$R_a R_{aba} R_{ababa} \cdots R_{bab} R_b = R_b R_{bab} \cdots R_{ababa} R_{aba} R_a. \tag{2.2}$$

In particular, if $a, b \in T$ and $ab = ba$, then (2.2) becomes $R_a R_b = R_b R_a$.

The collection $\{R_\tau\}_{\tau \in T}$ satisfying the Yang-Baxter equations (2.2) is sometimes called an (extensible) R-*matrix* (of the corresponding type).

Definition 2.1 makes sense for any (even infinite) Coxeter group. In the case of a Weyl group, equations (2.2), stated case-by-case in terms of the root system, were given by I. V. Cherednik (implicit in [7] and explicit in [8, Definition 2.1a]), along with a number of solutions. For example, the Yang-Baxter equation for a type A_2 dihedral subgroup with canonical generators s_α and s_β can be written in the form $R_{s_\alpha} R_{s_{\alpha+\beta}} R_{s_\beta} = R_{s_\beta} R_{s_{\alpha+\beta}} R_{s_\alpha}$. Notice that the equations (2.2) do not depend on the choice of the root system; thus the type B_n and type C_n systems of Yang-Baxter equations are exactly the same.

Remark 2.2. The above definition has a weaker version (cf. Cherednik [8, Definition 2.2]), in which we only ask for (2.2) to be satisfied for every *maximal* dihedral subgroup. (This distinction is only relevant in non-simply laced cases.) Although the stronger condition in Definition 2.1 is not needed to ensure that the general Yang-Baxter machinery works properly, it is actually satisfied by all solutions constructed in this paper.

For type A_{n-1}, the group W is the symmetric group S_n, the set T consists of all transpositions $(i\,j) \in S_n$, and the equations (2.2) are the celebrated (quantum) Yang-Baxter equations (see, e.g., [18]). Let us explain. Let R_{ij} be a shorthand for $R_{(i\,j)}$. Then (2.2) can be written as follows:

$$R_{ij} R_{kl} = R_{kl} R_{ij} \quad \text{if } i, j, k, l \text{ are distinct;} \tag{2.3}$$

$$R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij} \quad \text{if } i < j < k. \tag{2.4}$$

Example 2.3. The first solution to the Yang-Baxter equations (2.3)–(2.4) was given by C. N. Yang in his pioneering paper [26]. Specifically, Yang observed that the elements

$$R_{ij} = 1 + \frac{(i\,j)}{x_j - x_i} \tag{2.5}$$

in the group algebra of the symmetric group S_n satisfy (2.3)–(2.4), for any choice of distinct parameters $x_1, \dots, x_n$. This solution generalizes to an arbitrary Weyl group as follows (see [7], [8]). Let $\langle\ ,\ \rangle$ be the natural pairing of the vector spaces V and V^*. Choose a vector $x \in V^*$ so that $\langle x, \alpha\rangle \neq 0$ for any $\alpha \in \Phi$. Also let $(\varkappa_\alpha)_{\alpha\in\Phi}$ be a family of scalars invariant under the action of W on Φ. (In other words, the value of $\varkappa_\alpha$ only depends on whether the root α is short or long, assuming Φ is irreducible.) For any positive root α, the corresponding Yang's R-matrix is then given by

$$R_{s_\alpha} = 1 + \frac{\varkappa_\alpha\, s_\alpha}{\langle x, \alpha\rangle}. \tag{2.6}$$

We say that two reflection orderings φ and φ' of T are related by a *flip* if there is a maximal dihedral subgroup W' with canonical generators a and b such that

$$\varphi(a) = \varphi'(b),\ \ \varphi(aba) = \varphi'(bab), \dots,\ \ \varphi(bab) = \varphi'(aba),\ \ \varphi(b) = \varphi'(a),$$

and moreover, this sequence is an interval in $\{1, \dots, N\}$, and $\varphi(\tau) = \varphi'(\tau)$ for all reflections $\tau \notin W'$.

Lemma 2.4. Any two reflection orderings are related by a sequence of flips. □

Proof. Every two reduced decompositions of the element $w_o \in W$ are related by a sequence of elementary Coxeter transformations

$$s_1 s_2 \dots s_i \underbrace{s\,t\,s\,t\,s\dots}_{k} s_j \dots s_N \ \rightsquigarrow\ s_1 s_2 \dots s_i \underbrace{t\,s\,t\,s\,t\dots}_{k} s_j \dots s_N \tag{2.7}$$

(see, e.g., [17, Section 8.1]). The reflection orderings that correspond to two reduced decompositions in (2.7) via (2.1) are related by a flip with respect to the dihedral subgroup generated by $a = w^{-1}sw$ and $b = w^{-1}tw$, where $w = s_j \dots s_N$. This subgroup is maximal since it is conjugate to the maximal subgroup generated by simple reflections s and t. ■

Lemma 2.4 implies the following useful property of the solutions of the Yang-Baxter equations.

Proposition 2.5. Let $\{R_\tau\}$ be a solution of the Yang-Baxter equations for a finite Coxeter group W, and let $\varphi\colon\ T \to \{1, \dots, N\}$ be a reflection ordering. Then the product $R_{\varphi^{-1}(1)} \cdots R_{\varphi^{-1}(N)}$ does not depend on the choice of φ. □

3 Main result

Let k be a field of characteristic 0. Linear operators M_τ, $\tau \in T$, acting in the group algebra $k[W]$ of a Coxeter group W, are called *mixed Bruhat operators* if, for some families of scalars $\{p_\tau\}$ and $\{q_\tau\}$, we have

$$M_\tau(w) = \begin{cases} p_\tau \tau w & \text{if } \ell(\tau w) > \ell(w), \\ q_\tau \tau w & \text{if } \ell(\tau w) < \ell(w), \end{cases} \tag{3.1}$$

for $w \in W$. We are interested in finding solutions of the Yang-Baxter equations (2.2) that have the form

$$R_\tau = 1 + \varepsilon M_\tau, \tag{3.2}$$

where M_τ is given by (3.1), and ε is a formal variable. Although this problem is naturally stated for any Coxeter group, the family of solutions that we present below is constructed under the assumption that W is a Weyl group, and involves the associated root system Φ. Moreover, the solutions arising from root systems of types B and C (which correspond to the same Weyl group) are going to be different.

To describe our family of solutions of the Yang-Baxter equations (2.2) for a Weyl group W, we need the following notion. A k-valued function E defined on the set of positive roots Φ^+ is called *multiplicative* if, whenever $\alpha, \beta, \alpha+\beta \in \Phi^+$, we have

$$E(\alpha+\beta) = E(\alpha)\, E(\beta). \tag{3.3}$$

To construct such a function, simply assign arbitrary values to the simple roots, and then extend by multiplicativity. A typical example of a multiplicative function is given by

$$E(\alpha) = e^{\langle \alpha, x \rangle}, \tag{3.4}$$

where x is an arbitrary vector. Notice, however, that (3.4) does not allow for $E(\alpha) = 0$, a possibility that we do not want to exclude.

Theorem 3.1. Let W be a Weyl group with associated root system Φ. Then the operators R_τ given by (3.1)–(3.2) satisfy the quantum Yang-Baxter equations (2.2), provided the parameters p_τ and q_τ are defined by

$$p_{s_\alpha} = \frac{\kappa_\alpha E_1(\alpha)}{E_1(\alpha) - E_2(\alpha)}, \tag{3.5}$$

$$q_{s_\alpha} = \frac{\kappa_\alpha E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)}, \tag{3.6}$$

where

(i) E_1 and E_2 are any multiplicative functions on the set of positive roots Φ^+ such that $E_1(\alpha) \neq E_2(\alpha)$ for every $\alpha \in \Phi^+$, and

(ii) κ_α is a scalar whose value only depends on the length of $\alpha \in \Phi^+$. □

The proof of Theorem 3.1 is given in Section 5.

Remark 3.2. The generalized Yang's solution (2.6) can be obtained from the solution in Theorem 3.1 as a particular limiting case. Let $\kappa_\alpha = \delta\,\varkappa_\alpha$, where δ is a scalar. Fix a vector $x \in V^*$, and set $E_1(\alpha) = e^{\delta\langle\alpha,x\rangle}$ and $E_2(\alpha) = 1$. Making these substitutions into (3.5)–(3.6) and taking the limit as $\delta \to 0$, we obtain $p_{s_\alpha} = q_{s_\alpha} = \varkappa_\alpha/\langle\alpha, x\rangle$, which means that the operators $\lim_{\delta\to 0} M_{s_\alpha}$ act by left multiplication by $\varkappa_\alpha s_\alpha/\langle\alpha, x\rangle$, as desired.

4 Rescaling. Quantum Bruhat operators

For the rest of the paper, W is a Weyl group with the root system Φ. We denote by Φ^+ the set of positive roots.

In this section, we discuss rescaling, a very simple yet sometimes helpful way of producing new solutions to the Yang-Baxter equations from the existing ones. We show how rescaling of the mixed Bruhat operators leads in the limiting case to the construction of "quantum Bruhat operators" for an arbitrary Weyl group W. For type A, these operators are introduced and studied in [15]; the definition is motivated by the quantum Monk formula [14] (cf. below in this section). In this paper, we are mostly concerned with combinatorial applications of these operators.

Suppose that $\{M_\tau\}_{\tau\in T}$ is a family of mixed Bruhat operators such that the corresponding operators $R_\tau = 1 + \varepsilon M_\tau$ satisfy the Yang-Baxter equations (2.2). Let $\{\gamma_w\}_{w\in W}$ be a collection of nonzero scalars. Define the *rescaled* operators $\widetilde{M}_\tau$ by

$$\widetilde{M}_\tau(w) = \frac{\gamma_{\tau w}}{\gamma_w}\, M_\tau(w); \tag{4.1}$$

then the corresponding operators $\widetilde{R}_\tau = 1 + \varepsilon\widetilde{M}_\tau$ are again a solution to (2.2). This follows from the fact that $M_\tau(w)$ is always a scalar multiple of τw, and therefore $\widetilde{M}_\tau = \Gamma M_\tau \Gamma^{-1}$, where $\Gamma(v) = \gamma_v v$ for $v \in W$.

Let $\mathrm{ht}(\alpha)$ denote the *height* of a positive root α, i.e., the sum of the coefficients in the expansion of α in the basis of simple roots. We observe that for any scalar h, the function $\alpha \mapsto h^{\mathrm{ht}(\alpha)}$ is multiplicative.

Let $\delta \neq 0$ be a scalar parameter (eventually, we take $\delta \to 0$), and let $\alpha \mapsto E(\alpha)$ be a multiplicative function. Let the parameters $p_\tau = p_{s_\alpha}$ and $q_\tau = q_{s_\alpha}$ of the mixed Bruhat operators M_τ be given by (3.5)–(3.6) with

$$\begin{aligned} \kappa_\alpha &= \delta^{-1}, \\ E_1(\alpha) &= 1, \\ E_2(\alpha) &= \delta^{2\mathrm{ht}(\alpha)} E(\alpha). \end{aligned} \tag{4.2}$$

Using the notation $F \approx G$ for $\lim_{\delta \to 0} F/G = 1$, we then obtain the following:

$$\begin{aligned} p_{s_\alpha} &\approx \delta^{-1}, \\ q_{s_\alpha} &\approx \delta^{2\mathrm{ht}(\alpha)-1} E(\alpha). \end{aligned} \tag{4.3}$$

Now let the operators $\widetilde{M}_\tau$ be given by (4.1) with $\gamma_w = \delta^{\ell(w)}$. Then

$$\widetilde{M}_\tau(w) = \delta^{\ell(\tau w)-\ell(w)} M_\tau(w).$$

Combining this with (4.3) and (3.1) yields

$$\widetilde{M}_{s_\alpha}(w) \approx \begin{cases} \delta^{\ell(s_\alpha w)-\ell(w)-1}\, s_\alpha w & \text{if } \ell(s_\alpha w) > \ell(w)\,; \\ \delta^{\ell(s_\alpha w)-\ell(w)+2\mathrm{ht}(\alpha)-1} E(\alpha)\, s_\alpha w & \text{if } \ell(s_\alpha w) < \ell(w). \end{cases} \tag{4.4}$$

In what follows, we make use of the lemmas below.

Lemma 4.1. Any reflection τ has a symmetric reduced decomposition of the form $\tau = s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_1$. □

Proof. The proof is by induction on $\ell(\tau)$. Note that the length of any reflection is odd. The statement of the lemma is obviously true when $\ell(\tau) = 1$. If $\ell(\tau) \geq 3$, then find a simple reflection $s = s_\alpha \neq \tau$ such that $\ell(s\tau) = \ell(\tau s) < \ell(\tau)$. Then $0 > \tau(\alpha) \neq \alpha$, implying $s\tau(\alpha) < 0$. Hence $\ell(s\tau s) < \ell(s\tau) < \ell(\tau)$. By the inductive hypothesis, $s\tau s$ possesses a symmetric reduced decomposition. It remains to add s on both sides of it. ■

Lemma 4.2. Suppose that positive roots α and γ and a simple root β are such that $\alpha = s_\beta(\gamma)$ and $\ell(s_\alpha) > \ell(s_\gamma)$. Then $\alpha = m\beta + \gamma$, for a positive integer m. Moreover, $m = 1$ in the case of a simply laced root system. □

Proof. Note that $\ell(s_\alpha) > \ell(s_\gamma) \iff \ell(s_\gamma s_\beta) > \ell(s_\gamma) \iff s_\gamma(\beta) > 0$. It is therefore enough to verify the statement of the lemma for a rank 2 root system. (The only option in the simply laced case is A_2.) The verification is straightfoward. ■

Lemma 4.3. For any positive root α, we have $\ell(s_\alpha) \le 2\mathrm{ht}(\alpha) - 1$. In the case of a simply laced root system (the ADE case), $\ell(s_\alpha) = 2\mathrm{ht}(\alpha) - 1$ for any positive root α. □

Proof. Choose a symmetric reduced decomposition for s_α (cf. Lemma 4.1),

$$s_\alpha = s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_1,$$

and let $\beta_1, \ldots, \beta_k$ be the simple roots (not necessarily distinct) corresponding to the reflections $s_1, \ldots, s_k$, respectively. Repeatedly applying Lemma 4.2, we obtain

$$\begin{aligned}\alpha &= s_1 \cdots s_{k-1}(\beta_k) = s_1 \cdots s_{k-2}(m_{k-1}\beta_{k-1} + \beta_k) = \cdots \\ &= m_1\beta_1 + \cdots + m_{k-1}\beta_{k-1} + \beta_k,\end{aligned}$$

where the coefficients m_i are positive integers. (In the simply-laced case, $m_i = 1$ for all i.) Thus $\mathrm{ht}(\alpha) = (m_1 + \cdots + m_{k-1} + 1) \ge k$, while $\ell(s_\alpha) = 2k - 1$. ■

By Lemma 4.3, both exponents of δ in (4.4) are *nonnegative*. Letting $\delta \to 0$, we obtain the *quantum Bruhat operators* $Q_\tau = \lim_{\delta\to 0} \widetilde{M}_\tau$ given by

$$Q_{s_\alpha}(w) = \begin{cases} s_\alpha w & \text{if } \ell(s_\alpha w) = \ell(w) + 1; \\ E(\alpha)\, s_\alpha w & \text{if } \ell(s_\alpha w) = \ell(w) - 2\mathrm{ht}(\alpha) + 1; \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

In view of Lemma 4.3, the condition in the second line of (4.5) is equivalent to the pair of conditions $\ell(s_\alpha w) = \ell(w) - \ell(s_\alpha)$ and $\ell(s_\alpha) = 2\mathrm{ht}(\alpha) - 1$. The latter condition is redundant in the case of a simply laced root system.

Since the operators Q_τ were obtained from the mixed Bruhat operators of Theorem 3.1 by specializing parameters, rescaling, and taking a limit, we have arrived at the following result.

Corollary 4.4. Let $\{Q_\tau\}_{\tau\in T}$ be the quantum Bruhat operators defined by (4.5). Then the operators $R_\tau = 1 + \varepsilon Q_\tau$ satisfy the Yang-Baxter equations (2.2). □

For the type A case, it is noted in [15] that the operators Q_τ satisfy the *classical* Yang-Baxter equation, which is a slightly weaker statement than Corollary 4.4.

We now briefly explain the connection between our quantum Bruhat operators and the quantum cohomology of the generalized flag manifold G/B. Here G is a semisimple connected complex Lie group associated with the *dual* root system $\Phi^\vee$, and B is a Borel subgroup in G. For each element $w \in W$, let $[w]$ denote the *Schubert class*

$$[w] = \left[\overline{(Bw^{-1}B)/B}\right] \in H^{2\ell(w)}(G/B, \mathbb{Z}),$$

viewed as an element of the small quantum cohomology ring $QH^*(G/B, \mathbb{Z})$. (The reader is referred to [16], [14], and references therein for relevant background.)

Let $\mathbb{Z}[q]$ denote the polynomial ring generated by the deformation parameters of the quantum cohomology. The map $w \mapsto [w]$ extends to a linear isomorphism $\mathbb{Z}[q][W] \to QH^*(G/B)$; for this isomorphism, we use the same notation, $a \mapsto [a]$.

The Coxeter generators $s \in S$ correspond to the *special* Schubert classes $[s]$. Over $\mathbb{Z}[q]$, these classes generate the quantum cohomology ring.

Let the quantum Bruhat operators Q_{s_α} be given by (4.5), with the multiplicative function $\alpha \mapsto E(\alpha)$ defined by setting its values at simple roots to the corresponding deformation parameters of $QH^*(G/B)$. Then the "quantum Monk formula" for quantum multiplication of an arbitrary Schubert class $[w]$ by a special Schubert class $[s]$ can be written as

$$[w] * [s] = \sum_{\alpha>0} \langle \omega_s, \alpha \rangle \, [Q_{s_\alpha}(w)], \tag{4.6}$$

where ω_s denotes the fundamental weight corresponding to s. Equivalently,

$$[w] * [s] = \sum_{\substack{\alpha>0 \\ \ell(ws_\alpha)=\ell(w)+1}} \langle \omega_s, \alpha \rangle \, [ws_\alpha] + \sum_{\substack{\alpha>0 \\ \ell(ws_\alpha)=\ell(w)-2\mathrm{ht}(\alpha)+1}} \langle \omega_s, \alpha \rangle \, E(\alpha) \, [ws_\alpha]. \tag{4.7}$$

For the type A case, formula (4.7) is obtained in [14]. For a general type, it is given by D. Peterson (reproduced in [6], without proof).

5 Proof of Theorem 3.1

5.1 Cosets modulo dihedral reflection subgroups

As a prerequisite for our proofs, we need to understand the combinatorics of cosets modulo dihedral reflection subgroups, viewed as subposets of the Bruhat order. The following statement is known to hold for any Coxeter group (see Dyer [11]); in the special case of a Weyl group, it has a simple proof provided below.

Lemma 5.1. Let W' be a dihedral reflection subgroup of W, and let S' be its set of canonical generators. Then the Bruhat order on W' (viewed as a Coxeter group with generating set S') coincides with the partial order induced from the Bruhat order on W.

The Bruhat order on W' is isomorphic to the partial order on any coset $W'w$ (induced from the Bruhat order on W). More precisely, $W'w$ has a unique minimal element $\tilde{w}$, and the map $w' \mapsto w'\tilde{w}$ is an isomorphism of posets W' and $W'w = W'\tilde{w}$. □

The last statement can be rephrased as saying that for any $w' \in W'$ and $\tau \in T \cap W'$, we have $\ell(\tau w') < \ell(w')$ if and only if $\ell(\tau w'\tilde{w}) < \ell(w'\tilde{w})$.

Proof. For a positive root α and $w \in W$, the condition $\ell(s_\alpha w) < \ell(w)$ is equivalent to $w^{-1}(\alpha) < 0$. Since the same criterion describes the covering relation in W', the first part of the lemma follows. To prove the second part, choose $\tilde{w}$ to be the element of minimal length in $W'w$ (if there are several such, pick any). Take any positive root α such that $s_\alpha \in T \cap W'$. Then $\ell(s_\alpha \tilde{w}) > \ell(\tilde{w})$ and therefore $\tilde{w}^{-1}(\alpha) > 0$. Thus $\tilde{w}^{-1}$ maps every positive root that corresponds to a reflection in W' into a positive root (and every negative into a negative). Hence

$$\ell(s_\alpha w') < \ell(w') \Longleftrightarrow (w')^{-1}(\alpha) < 0 \Longleftrightarrow \tilde{w}^{-1}(w')^{-1}(\alpha) < 0 \Longleftrightarrow \ell(s_\alpha w'\tilde{w}) < \ell(w'\tilde{w}),$$

proving the last claim of the lemma. It then follows that $\tilde{w}$ is indeed the unique minimal element of the coset $W'w$. ■

Let W' be a dihedral reflection subgroup of W. (Thus W' is of type $A_1 \times A_1$, A_2, B_2, or G_2.) Our goal is to check the Yang-Baxter equation (2.2) corresponding to W' for the mixed Bruhat operators R_τ, assuming some special choice of parameters p_τ and q_τ. Notice that for *any* choice of parameters, the operators involved in that particular equation leave each subspace $k[W'w]$ (the span of the right coset $W'w$) invariant. Thus the operators R_τ satisfy the equation in question if and only if it is satisfied by the restrictions of these operators onto each of these invariant subspaces. In fact, it is enough to just consider the restrictions onto $k[W']$ for the following reason: the second part of Lemma 5.1 implies that for a coset $W'w$ with the minimal element $\tilde{w}$, the linear isomorphism $k[W'] \to k[W'w]$ defined by $w' \mapsto w'\tilde{w}$ intertwines the actions of the operators R_τ on $k[W']$ and $k[W'w]$, respectively.

Thus the verification of the Yang-Baxter equation associated with W' for the mixed Bruhat operators R_τ boils down to verifying this equation for the restrictions of the operators participating in this particular equation onto the invariant subspace $k[W']$ (which has dimension 4, 6, 8, or 12). The action of these operators on $k[W']$ is in turn determined by the first part of Lemma 5.1: they act on $k[W']$ as if W' were the whole Weyl group.

The case of W' of type $A_1 \times A_1$ is easily verified: one checks directly that, for any choice of parameters, the mixed Bruhat operators R_τ and R_σ commute whenever the reflections τ and σ do. Thus we only need to take care of (2.2) in the cases where both sides involve at least three factors.

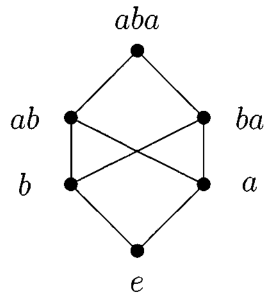

Figure 1 The Bruhat order on a reflection subgroup of type A_2

Let us summarize.

Proposition 5.2. The mixed Bruhat operators R_τ satisfy the Yang-Baxter equations (2.2) if and only if, for any reflection subgroup $W' \subset W$ of type A_2, B_2, or G_2, the corresponding Yang-Baxter equation is satisfied by the restrictions of the operators involved in that equation onto the invariant subspace $k[W']$. □

Thus to prove Theorem 3.1, it remains to do so for W of types A_2, B_2, and G_2.

5.2 Type A_2

Suppose that W is the symmetric group S_3, and let a and b be its canonical Coxeter generators. The Bruhat order on W is given in Figure 1.

We should verify that the operators R_τ defined by (3.1)–(3.2) satisfy the quantum Yang-Baxter equation

$$R_a R_{aba} R_b = R_b R_{aba} R_a \tag{5.1}$$

(cf. (2.4)), provided the parameters p_τ and q_τ are given by (3.5)–(3.6). Substituting $R_\tau = 1 + \varepsilon M_\tau$, we observe that the terms of degrees 0 and 1 in ε are clearly the same on both sides of (5.1). Equating the quadratic terms gives the *classical Yang-Baxter equation* (see [18])

$$[M_a, M_b] = [M_b, M_{aba}] + [M_{aba}, M_a] \tag{5.2}$$

(here $[A, B] = AB - BA$ stands for the commutator), while equating the cubic terms gives the quantum Yang-Baxter equation for the M_τ:

$$M_a M_{aba} M_b = M_b M_{aba} M_a \quad \text{if } i < j < k. \tag{5.3}$$

Thus we have to check (5.2) and (5.3).

In the special case under consideration, the operators M_a, M_b, and $M_{aba} = M_{bab}$ are readily computed from the definition (3.1). Specifically, in the linear basis of $k[W]$ formed by the elements e, a, b, ab, ba, aba (in this order), they are given by the following matrices:

$$M_a = \begin{bmatrix} 0 & q_a & 0 & 0 & 0 & 0 \\ p_a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_a & 0 & 0 \\ 0 & 0 & p_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_a \\ 0 & 0 & 0 & 0 & p_a & 0 \end{bmatrix}, \quad M_b = \begin{bmatrix} 0 & 0 & q_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_b & 0 \\ p_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_b \\ 0 & p_b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_b & 0 & 0 \end{bmatrix},$$
$$M_{aba} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & q_{aba} \\ 0 & 0 & 0 & q_{aba} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{aba} & 0 \\ 0 & p_{aba} & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{aba} & 0 & 0 & 0 \\ p_{aba} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.4}$$

Substituting (5.4) into (5.2), we obtain, upon simplifications, the following system of equations:

$$\begin{cases} -q_a q_b + p_b q_{aba} + q_{aba} q_a &= 0; \\ q_a q_b - q_b q_{aba} - q_{aba} p_a &= 0; \\ p_a q_b - q_b p_{aba} - q_{aba} p_a &= 0; \\ -q_a p_b + p_b q_{aba} + p_{aba} q_a &= 0; \\ p_a p_b - q_b p_{aba} - p_{aba} p_a &= 0; \\ -p_a p_b + p_b p_{aba} + p_{aba} q_a &= 0. \end{cases} \tag{5.5}$$

Making the same substitution into (5.3), we obtain a single equation $q_a p_{aba} q_b = p_a q_{aba} p_b$, which actually follows from (5.5); indeed, multiply the first equation in (5.5) by p_{aba}, and subtract the last one, multiplied by q_{aba}.

It remains to check that (3.5)–(3.6) imply (5.5). First we note that all roots have the same length, so we may drop the subscript α in κ_α. Second, subtracting (3.6) from (3.5) yields

$$p_\tau = q_\tau + \kappa. \tag{5.6}$$

When we substitute $p_a = q_a + \kappa$, $p_b = q_b + \kappa$, and $p_{aba} = q_{aba} + \kappa$ into the system of equations (5.5), it reduces to the single equation

$$q_{aba}(q_a + q_b + \kappa) = q_a q_b. \tag{5.7}$$

Let α and β be the positive roots corresponding to a and b, respectively. Then (5.7) becomes $q_{s_{\alpha+\beta}}(q_{s_\alpha} + q_{s_\beta} + \kappa) = q_{s_\alpha} q_{s_\beta}$. Substituting (3.6) into this equation, we obtain

$$\frac{E_2(\alpha+\beta)}{E_1(\alpha+\beta) - E_2(\alpha+\beta)} \left(\frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} + \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} + 1 \right)$$
$$= \frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)},$$

which is routinely checked using the fact that E_1 and E_2 are multiplicative (cf. (3.3)).

This completes the proof of Theorem 3.1 for the type A_2 case and therefore for any simply laced type.

It is possible to use equations (5.5) to provide a complete (albeit cumbersome) parametric description of all solutions of the Yang-Baxter equations of, say, type A. On the other hand, once we impose some relatively weak "nondegeneracy" restrictions on the parameters q_τ, the solution given by (3.5)–(3.6) becomes exhaustive. We state the corresponding result below, omitting a straightforward proof.

Proposition 5.3. Let W be a Weyl group of type A, D, or E. Let $\{q_\tau\}_{\tau \in T}$ be a family of scalar parameters, let $\kappa \in k$ be a constant, and let $p_\tau = q_\tau + \kappa$. Then the following are equivalent.

(a) The mixed Bruhat operators R_τ defined by (3.1)–(3.2) satisfy the Yang-Baxter equations (2.2).

(b) For any reflection subgroup $W' \subset W$ of type A_2 with canonical generators a and b, we have (5.7).

(c) The parameters q_τ are given by (3.6), with the multiplicative functions E_1 and E_2 defined by setting $E_1(\alpha) = p_{s_\alpha}$ and $E_2(\alpha) = q_{s_\alpha}$ for every *simple* root α. □

5.3 Types B_2 and G_2

The proof in these cases is similar to the type A_2 case. We begin by observing that formulas (3.5)–(3.6) imply

$$p_{s_\gamma} = q_{s_\gamma} + \kappa_\gamma, \quad \gamma \in \Phi^+; \tag{5.8}$$

recall that κ_γ only depends on whether root γ is short or long (cf. (5.6)).

Let a and b be the canonical generators of W. If W is of type B_2, then $T = \{a, aba, bab, b\}$. Using the elements e, a, b, ba, ab, aba, bab, $abab$ (in this order) as a basis for $k[W]$, we obtain the matrices

$$M_a = \begin{bmatrix} 0 & q_a & 0 & 0 & 0 & 0 & 0 & 0 \\ p_a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_a & 0 & 0 \\ 0 & 0 & p_a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_a \\ 0 & 0 & 0 & 0 & 0 & 0 & p_a & 0 \end{bmatrix}, \quad M_b = \begin{bmatrix} 0 & 0 & q_b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_b & 0 & 0 & 0 & 0 \\ p_b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_b \\ 0 & 0 & 0 & 0 & p_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_b & 0 & 0 \end{bmatrix},$$

and, in a similar way, the matrices M_{aba} and M_{bab}. Substituting these matrices into the type B_2 Yang-Baxter equation

$$(1 + M_a)(1 + M_{aba})(1 + M_{bab})(1 + M_b) = (1 + M_b)(1 + M_{bab})(1 + M_{aba})(1 + M_a),$$

we obtain a system of equations for the 8 parameters p_τ and q_τ corresponding to $\tau \in T$. Once we make the substitution (5.8), this system of equations collapses into the single equation

$$q_a q_b = q_a q_{aba} + q_{aba} q_{bab} + q_{bab} q_b + \kappa_\alpha q_{aba} + \kappa_\beta q_{bab}, \tag{5.9}$$

where, as before, α and β are the positive roots corresponding to a and b, respectively. Note that equation (5.9) is invariant under the transformation that interchanges a and b while simultaneously interchanging α and β. Therefore, without loss of generality, we may assume that α is short and β is long. Then aba and bab correspond to positive roots $2\alpha + \beta$ (long) and $\alpha + \beta$ (short), respectively. Substituting (3.6) into (5.9) and factoring out $\kappa_\alpha \kappa_\beta$, we obtain

$$\begin{aligned} &\frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} \\ &\quad = \frac{E_2(\alpha)}{E_1(\alpha) - E_2(\alpha)} \frac{E_2(2\alpha + \beta)}{E_1(2\alpha + \beta) - E_2(2\alpha + \beta)} \\ &\qquad + \frac{E_2(2\alpha + \beta)}{E_1(2\alpha + \beta) - E_2(2\alpha + \beta)} \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)} \\ &\qquad + \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)} \frac{E_2(\beta)}{E_1(\beta) - E_2(\beta)} \\ &\qquad + \frac{E_2(2\alpha + \beta)}{E_1(2\alpha + \beta) - E_2(2\alpha + \beta)} + \frac{E_2(\alpha + \beta)}{E_1(\alpha + \beta) - E_2(\alpha + \beta)}, \end{aligned}$$

which is easily checked using (3.3).

For the type G_2 case, we use (5.8) to express everything in terms of the 6 parameters q_τ and the 2 parameters κ_γ (for the short and long roots, respectively). Making a substitution into the Yang-Baxter equation of type G_2, we obtain two equations,

$$\begin{aligned} q_a q_b = {} & q_a q_{aba} + q_{aba} q_{ababa} + q_{ababa} q_{babab} + q_{babab} q_{bab} + q_{bab} q_b \\ & + \kappa_\alpha q_{aba} + \kappa_\alpha q_{babab} + \kappa_\beta q_{bab} + \kappa_\beta q_{ababa} \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} & - q_a q_{bab} + q_a q_{ababa} - q_b q_{aba} + q_b q_{babab} + q_{aba} q_{babab} + q_{bab} q_{ababa} \\ & + q_a q_b q_{aba} q_{bab} - q_a q_b q_{aba} q_{ababa} - q_a q_b q_{bab} q_{babab} - q_a q_b q_{ababa} q_{babab} \\ & + q_a q_{aba} q_{bab} q_{babab} + q_b q_{bab} q_{ababa} q_{babab} + q_b q_{aba} q_{bab} q_{ababa} \\ & + q_a q_{aba} q_{ababa} q_{babab} + q_{aba} q_{bab} q_{ababa} q_{babab} \\ & + \kappa_\alpha \big(q_{ababa} - q_a q_b q_{babab} + q_b q_{aba} q_{bab} + q_a q_{aba} q_{babab} \\ & \qquad + q_{aba} q_{bab} q_{babab} + q_{aba} q_{ababa} q_{babab} \big) \\ & + \kappa_\beta \big(q_{babab} - q_a q_b q_{ababa} + q_a q_{aba} q_{bab} + q_b q_{bab} q_{ababa} \\ & \qquad + q_{aba} q_{bab} q_{ababa} + q_{bab} q_{ababa} q_{babab} \big) \\ & + \kappa_\alpha^2 q_{aba} q_{babab} + \kappa_\alpha \kappa_\beta q_{aba} q_{bab} + \kappa_\beta^2 q_{bab} q_{ababa} = 0. \end{aligned} \tag{5.11}$$

Similarly to cases A_2 and B_2 , we then verify these identities using the substitution (3.6) (preferably with the help of a computer). ∎

6 Tilted Bruhat orders

We now apply the results of Section 4 to the combinatorics of the Weyl group W. The following definition is motivated by the formula (4.5) for the quantum Bruhat operators. As before, Φ is a root system associated with W, and Φ^+ denotes the set of positive roots.

Definition 6.1. Let D_Φ denote the directed graph whose vertices are the elements of W, and whose edges are of the form $(u, s_\alpha u)$, where $u \in W$, $\alpha \in \Phi^+$, and either $\ell(s_\alpha u) = \ell(u)+1$ or $\ell(s_\alpha u) = \ell(u) - 2\mathrm{ht}(\alpha) + 1$.

Note that the digraph D_Φ depends on the root system Φ (and on the choice of the positive system $\Phi^+ \subset \Phi$), not just on the Weyl group W. Thus, for example, digraphs of types B_n and C_n corresponding to dual root systems with the same Weyl group are different.

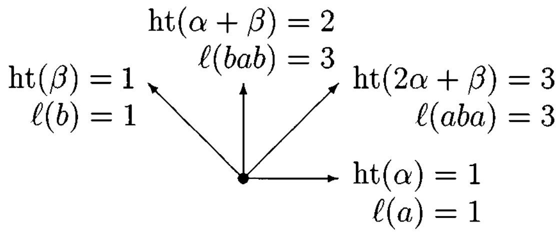

Figure 2 Root system B_2

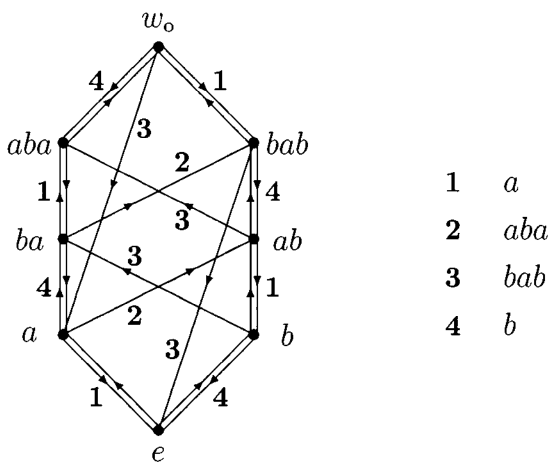

Figure 3 The digraph D_Φ for W of type B_2

Because of Lemma 4.3, the condition $\ell(s_\alpha u) = \ell(u) - 2\mathrm{ht}(\alpha) + 1$ in Definition 6.1 is equivalent to $\ell(u) - \ell(s_\alpha u) = \ell(s_\alpha) = 2\mathrm{ht}(\alpha) - 1$. We also remark that D_Φ is the directed graph with vertex set W whose adjacency matrix is the matrix of the operator $\sum_\tau Q_\tau$, where the Q_τ are given by (4.5) with $E(\alpha) = 1$ for all α.

Once a reflection ordering φ for W is chosen, we label the edges of D_Φ by assigning the label $\varphi(\tau)$ to an edge (u, v) with $v = \tau u$. We write $u \xrightarrow{m} v$ to denote that (u, v) is an edge in D_Φ labelled by m.

Example 6.2. Consider a Weyl group of type B_2. This is the first instance where the condition $\ell(s_\alpha) = 2\mathrm{ht}(\alpha) - 1$ comes into play. Let a and b be the generators of W that correspond to the simple roots α (short) and β (long), respectively. Then the reflections a, b, and bab satisfy this condition, while aba does not (see Figure 2). The resulting graph D_Φ, for the reflection ordering $a < aba < bab < b$, is shown in Figure 3. Note that we disallow down-directed edges that correspond to multiplying by aba (on the left).

Definition 6.3. For $u, v \in W$, let $\ell(u, v)$ denote the length of the shortest directed path in D_Φ from u to v. In particular, $v \mapsto \ell(e, v) = \ell(v)$ is the usual length function; moreover,

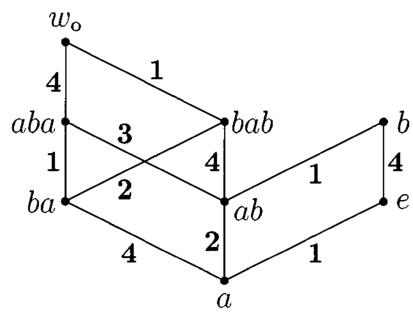

Figure 4 Tilted Bruhat order $B_\Phi(a)$ for W of type B_2

$\ell(u,v) = \ell(v) - \ell(u)$ whenever $u \le v$ in the Bruhat order. The *tilted Bruhat order* $B_\Phi(u)$ is the graded partial order on W defined by

$$w_1 \preceq_u w_2 \overset{\text{def}}{\Longleftrightarrow} \ell(u, w_1) + \ell(w_1, w_2) = \ell(u, w_2).$$

In other words, $w_1 \preceq_u w_2$ if and only if there exists a shortest path in D_Φ from u to w_2 that passes through w_1.

We note that $B_\Phi(e)$ is the usual Bruhat order on W.

Any choice of reflection ordering induces edge labelling of the Hasse diagram of $B_\Phi(u)$ inherited from D_Φ. Figure 4 shows an example of a tilted Bruhat order which is not pure (i.e., does not have a unique maximal element $\hat{1}$); here W is of type B_2, and we use the same conventions as in Figure 3.

A nonempty interval $[w_1, w_2]$ in $B_\Phi(u)$ is called a *tilted Bruhat interval* between w_1 and w_2. This interval does not depend on the choice of u (as long as $w_1 \preceq_u w_2$), but on w_1 and w_2 alone, since

$$[w_1, w_2] = \{w \in W\colon\ \ell(w_1, w) + \ell(w, w_2) = \ell(w_1, w_2)\}.$$

Moreover, the induced partial order on $[w_1, w_2]$ is independent of the choice of u, as long as $w_1 \preceq_u w_2$. More precisely, $[w_1, w_2]$ is the graded poset whose Hasse diagram is the minimal subgraph of D_Φ containing all directed paths from w_1 to w_2 that have the smallest possible length. Note that this definition applies for *any* pair $w_1, w_2 \in W$. In the special case where $w_1 \le w_2$ (in the ordinary Bruhat order), the notion of a tilted Bruhat interval specializes to the usual notion of an interval in the Bruhat order.

Figure 5 shows the tilted Bruhat intervals $[ab, a]$ and $[w_o, e]$ for W of type B_2, with the same notation as before. (The first of these intervals coincides with the tilted Bruhat order $B_\Phi(ab)$.) Notice that $[w_o, e]$ is by no means dual to $[e, w_o] = W$.

Our main combinatorial result is an extension of a certain fundamental property of Bruhat orders to their "tilted analogues" introduced in Definition 6.3.

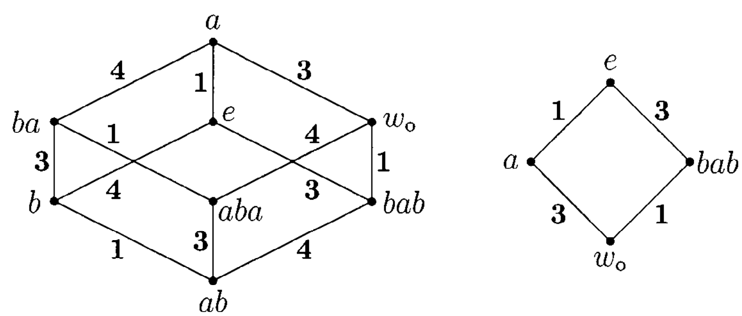

Figure 5 Tilted Bruhat intervals $[ab, a]$ and $[w_\circ, e]$

Theorem 6.4. Fix a reflection ordering φ in a Weyl group W.

(1) For any pair of elements $u, v \in W$, there is a unique path from u to v in the directed graph D_Φ such that its sequence of labels is strictly increasing (resp., strictly decreasing).

(2) The unique label-increasing (resp., label-decreasing) path from u to v in D_Φ has the smallest possible length $\ell(u, v)$. Moreover, it is lexicographically minimal (resp., lexicographically maximal) among all shortest paths from u to v. $\square$

The proof of Theorem 6.4 is given at the end of this section.

In the terminology of [3], Theorem 6.4 asserts that the tilted Bruhat order (hence any tilted Bruhat interval $[u, v]$) is *EL-shellable* (hence *CL-shellable*), with the EL-shelling provided by any reflection ordering (and therefore by its reversal as well). In the case where $u = e$, Theorem 6.4 specializes to the EL-shellability theorem for the ordinary Bruhat orders, a result conjectured by Björner and proved by Dyer [13, Proposition 4.3]. (Dyer's theorem holds for an arbitrary Coxeter group, with an appropriate generalization of the concept of a reflection ordering.)

Shellability can be used to compute the *Möbius function* [21] of the Bruhat order (ordinary or tilted). Recall (see [22]) that a finite graded poset with $\hat{0}$ and $\hat{1}$ (resp., with $\hat{0}$) is called *Eulerian* (resp., *lower Eulerian*) if its Möbius function is given by

$$\mu(x, y) = (-1)^{\mathrm{rank}(y) - \mathrm{rank}(x)}$$

for any $x \leq y$. This is equivalent to requiring that any interval $[x, y]$ contains an equal number of elements of even and odd rank.

A well-known theorem of Verma [24], [25] (cf. also [2], [9], [19]) asserts that any interval in the Bruhat order of any Coxeter group is Eulerian. To our knowledge, no simple proof of this result is known, except for the special case $x = e$ (see Lascoux [20, Lemma 1.13]). The story of Verma's theorem is told in [17, p. 176]. Remarkably, it can be strengthened as follows: any Bruhat interval is actually a face poset of a shellable regular CW sphere (see Björner [1, Theorem 5.1] and Björner and Wachs [2, Theorem 4.2]); hence it is also Cohen-Macaulay (see [4]).

All the statements mentioned in the preceding paragraph are implied by Dyer's shellability theorem [13, Proposition 4.3]. Not surprisingly, our Theorem 6.4 can be used to extend these results to the tilted case.

Corollary 6.5. Each tilted Bruhat order $B_\Phi(u)$ of a Weyl group W is a lexicographically shellable lower Eulerian poset. As a consequence, any tilted Bruhat interval is a face poset of a shellable regular CW sphere. In particular, it is Eulerian and Cohen-Macaulay. □

Proof. By [1, Proposition 4.5], Theorem 6.4 implies that $[u, v]$ is a face poset of a regular CW sphere. Such posets are known to be both Eulerian and Cohen-Macaulay; see, e.g., Stanley [22, Section 1]. ■

The Eulerian property can also be deduced directly from Theorem 6.4, as follows. By a simple counting argument (cf. [4, Corollary 2.3]), the values of the Möbius function can be computed from an EL-shelling by

$$\mu(u, v) = (-1)^{\mathrm{rank}(y)-\mathrm{rank}(x)} \cdot \text{(number of label-decreasing chains from } u \text{ to } v\text{)}.$$

In our case, there is exactly one such chain, and the Eulerian property follows.

Proof of Theorem 6.4. We first prove Part 1 of the theorem. As before, φ is a reflection ordering, $N = \ell(w_o)$, $R_\tau = 1 + \varepsilon Q_\tau$ for $\tau \in T$, and the Q_τ are given by (4.5) with $E(\alpha) = 1$.

We eventually prove the following statement, which can be viewed as an algebraic reformulation of Theorem 6.4 (excluding the last statement of the latter).

Proposition 6.6. For any $u \in W$,

$$R_{\varphi^{-1}(1)} \cdots R_{\varphi^{-1}(N)}(u) = \sum_{v \in W} \varepsilon^{\ell(u,v)}\, v. \tag{6.1}$$

□

Part 1 of Theorem 6.4 is equivalent to the special case $\varepsilon = 1$ of (6.1), for the following reasons. We have already noted (see comments following Definition 6.1) that (u, v) is an edge in D_Φ if and only if $v = Q_\tau(u)$ for some $\tau \in T$, in which case (u, v) is labelled by $\varphi(\tau)$. Thus the identity (6.1), with $\varepsilon = 1$, asserts existence and uniqueness of the label-decreasing path.

Let us denote by $\mathcal{T}$ the specialization of the operator $R_{\varphi^{-1}(1)} \cdots R_{\varphi^{-1}(N)}$ obtained by setting $\varepsilon = 1$. By Proposition 2.5 and Corollary 4.4, the operator $\mathcal{T}$ does not depend on the choice of reflection ordering φ.

We identify an element $w \in W$ with the linear operator $u \mapsto wu$ in $k[W]$. Let $s \in S$. Then (4.5), with $\varepsilon = 1$, gives $Q_s = s$, implying

$$(1 + Q_s)s = 1 + Q_s. \tag{6.2}$$

Since there exists a reduced decomposition of w_o that ends in s, there also exists a reflection ordering φ such that $\varphi^{-1}(N) = s$ (cf. (2.1)). Hence (6.2) implies that

$$\mathcal{T}s = \left(\prod_{i=1}^{N-1}(1 + Q_{\varphi^{-1}(i)})\right)(1 + Q_s)\, s = \mathcal{T}.$$

It follows that, more generally, $\mathcal{T}w = \mathcal{T}$ for all $w \in W$. Analogously, one shows that $w\mathcal{T} = \mathcal{T}$ for all $w \in W$. These equations can be interpreted as saying that the matrix of $\mathcal{T}$ in the basis W of $k[W]$ is invariant under permutations of rows and columns. Hence there exists a constant c such that, for any $u \in W$,

$$\mathcal{T}(u) = c \sum_{v \in W} v.$$

On the other hand, it is clear from the definition of the Q_τ that the coefficient of w_o in $\mathcal{T}(e)$ is ≤ 1, where $e \in W$ is the identity element. Since c is obviously a positive integer, we conclude that $c = 1$, and part (1) is proved.

To prove the rest, we need the following lemma, which generalizes the corresponding result for the ordinary Bruhat order (see, e.g., [13, Lemma 4.1]).

Lemma 6.7. Assume that

$$u, x, v \in W, \quad u \xrightarrow{k} x \xrightarrow{l} v, \quad k > l. \tag{6.3}$$

Then there exists a unique $y \in W$ such that (cf. Figure 6)

$$u \xrightarrow{m} y \xrightarrow{n} v, \quad l < n > m < k. \tag{6.4}$$

□

Proof. Consider the dihedral group W' generated by the reflections $\varphi^{-1}(k)$ and $\varphi^{-1}(l)$. Define the operators Q_τ and R_τ as in Theorem 6.4, and write down the Yang-Baxter equation (2.2) for W', so that the order of the terms in the left-hand side was compatible with the reflection order φ. Thus the sequence of reflections appearing on the left-hand side

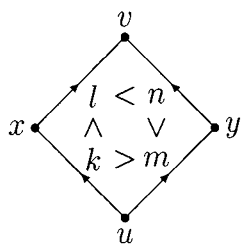

Figure 6 Tilted Bruhat interval of length 2

is label-increasing, while the one on the right-hand side is label-decreasing. Apply the left-hand side to u, and take the coefficient of $\varepsilon^2 v$. This is the number of label-decreasing paths in D_Φ from u to v that have length 2 and stay within the coset $W'u$. We know one such path, namely, $u \xrightarrow{k} x \xrightarrow{l} v$. By the Yang-Baxter equation, there should also be a label-increasing path of length 2 from u to v that stays within $W'u$; let us denote it by $u \xrightarrow{m} y \xrightarrow{n} v$. It remains to check that $k > m$ and $l < n$. These two statements are completely analogous to each other, so we only show how to prove the first one. Suppose that, on the contrary, $k < m$. (Obviously, $k \neq m$, since otherwise $x = y$.) Then $l < k < m < n$, which in particular means that the four reflections labelled by l, k, m, n are all distinct. If W' is of type A_2, then this already brings the desired contradiction, since in that case, there are only three reflections in W'. If W' is of type B_2, with canonical generators a and b (say, $\varphi(a) < \varphi(b)$, then there are four reflections in W', and therefore l, k, m, n correspond to a, aba, bab, b, respectively. But this implies that $v = a \cdot aba \cdot u = b \cdot bab \cdot u$, a contradiction. The remaining case $W' = W = G_2$ is checked directly. ■

We can now complete the proof of Theorem 6.4 using an argument borrowed from [13, p. 114]. Among all shortest paths in D_Φ from u to v, let

$$u = w_1 \to w_2 \to \cdots \to w_d = v \tag{6.5}$$

be the one whose label sequence is lexicographically minimal. To prove part (2) of the theorem, we need to show that this path is label-increasing. Suppose otherwise; i.e., for some $i \in \{2, \ldots, d-1\}$, we have $w_{i-1} \xrightarrow{k} w_i \xrightarrow{l} w_{i+1}$ with $k > l$. (We cannot have $k = l$ since this would create a loop, and the path would not be shortest.) Then, by Lemma 6.7, there exists $y \in W$ such that $w_{i-1} \xrightarrow{m} y \xrightarrow{n} w_{i+1}$ and $m < k$. Thus, replacing w_i by y in (6.5) produces a chain with a lexicographically smaller sequence of labels—a contradiction. ■

It remains to observe that Proposition 6.6 is a direct corollary of Theorem 6.4.

Acknowledgments

We are grateful to Andrei Zelevinsky for his fair and constructive criticism that brought about a substantial improvement over the original version. We also thank Anders Björner, Alain Lascoux, and Richard Stanley for useful comments and conversations, and Jenny Levine at the IMRN editorial office for her careful editing. Part of our work was carried out while the first two authors were participating in the "Combinatorics" program at MSRI (Mathematical Sciences Research Institute).

The authors were supported in part by MSRI. Sergey Fomin was also supported by National Science Foundation grant #DMS-9700927.

References

[1] A. Björner, *Posets, regular CW complexes and Bruhat order*, European J. Combin. **5** (1984), 7–16.

[2] A. Björner and M. Wachs, *Bruhat order of Coxeter groups and shellability*, Adv. in Math. **43** (1982), 87–100.

[3] ———, *On lexicographically shellable posets*, Trans. Amer. Math. Soc. **277** (1983), 323–341.

[4] A. Björner, A. M. Garsia, and R. P. Stanley, "An introduction to Cohen-Macaulay partially ordered sets" in *Ordered Sets (Banff, Alta, 1981)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **83**, ed. I. Rival, Reidel, Dordrecht, 1982, 583–615.

[5] N. Bourbaki, *Éléments de Mathématique: Groupes et algèbres de Lie*, Ch. IV–VI, Hermann, Paris, 1968.

[6] J. Carrell, *Quantum cohomology of* G/P, notes from three lectures of Dale Peterson, preprint, 1996.

[7] I. V. Cherednik, *Factorizing particles on a half line, and root systems*, Teoret. Mat. Fiz. **61** (1984), 35–44.

[8] ———, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Comm. Math. Phys. **150** (1992), 109–136.

[9] V. V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math. **39** (1977), 187–198.

[10] ———, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. **53** (1989), 543–546.

[11] M. Dyer, *Reflection subgroups of Coxeter systems*, J. Algebra **135** (1990), 57–73.

[12] ———, *On the "Bruhat graph" of a Coxeter system*, Compositio. Math. **78** (1991), 185–191.

[13] ———, *Hecke algebras and shellings of Bruhat intervals*, Compositio. Math. **89** (1993), 91–115.

[14] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc. **10** (1997), 565–596.

[15] S. Fomin and A. N. Kirillov, "Quadratic algebras, Dunkl elements, and Schubert calculus" in *Advances in Geometry*, ed. J. L. Brylinski, et al, Progr. Math. **172**, Birkhäuser, Boston, 1999, 147–182.

[16] W. Fulton and R. Pandharipande, "Notes on stable maps and quantum cohomology" in *Algebraic Geometry (Santa Cruz, CA, 1995)*, Proc. Sympos. Pure Math. **62**, Amer. Math. Soc., Providence, 1997, 45–96.

[17] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Stud. Adv. Math. **29**, Cambridge Univ. Press, Cambridge, 1990.

[18] M. Jimbo, editor, *Yang-Baxter equation in integrable systems*, Adv. Ser. Math. Phys. **10**, World Scientific, Teaneck, NJ, 1989.

[19] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[20] A. Lascoux, "Anneau de Grothendieck de la variété de drapeaux" in *The Grothendieck Festschrift*, *Vol. III*, Progr. Math. **88**, Birkhäuser, Boston, 1990, 1–34.

[21] G.-C. Rota, "On the foundations of combinatorial theory I. Theory of Möbius functions" in *Gian-Carlo Rota on combinatorics*, ed. P. S. Kung, Contemp. Math., Birkhäuser, Boston, 1995, 3–31.

[22] R. P. Stanley, "A survey of Eulerian posets" in *Polytopes: Abstract*, *Convex and Computational (Scarborough*, *ON*, *1993)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **440**, Kluwer Acad. Publ., Dordrecht, 1994, 301–333.

[23] ———, *Enumerative Combinatorics*, *Vol. 1*, 2d ed., Cambridge Stud. Adv. Math. **49**, Cambridge Univ. Press, Cambridge, 1997.

[24] D.-N. Verma, *Möbius inversion for the Bruhat order on a Weyl group*, Ann. Sci. École Norm. Sup. **4** (1971), 393–398.

[25] ———, *A strengthening of the exchange property of Coxeter groups*, preprint, 1972.

[26] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967), 1312–1315.

Brenti: Dipartimento di Matematica, Universitá di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy; brenti@mat.uniroma2.it

Fomin: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA; fomin@math.mit.edu

Postnikov: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA; apost@math.mit.edu