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RESEARCH
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ON GENERALIZATIONS OF ABEL'S AND HURWITZ'S IDENTITIES

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RRR 48-99, December, 1999

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Alexander Kelmans   Alexander Postnikov

Abstract. In 1826 N. Abel found a generalization of the binomial formula. In 1902 Abel's theorem has been further generalized by A. Hurwitz. In this paper we give combinatorial interpretations of Abel's and Hurwitz's identities. Moreover we describe a mechanism that provides infinitely many identities each being a generalization of Hurwitz's identity.

Contents

1 Introduction

In the 19-th century, N. Abel found the following surprising generalization of the binomial formula [1] (see also [5, 6]):

1.1 $(x+y)^n = \sum\{\binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k} : k \in [0,n]\}$.

Here and below $[k,s] = \{k,\ldots,s\}$ where k and s are integers and $k \le s$.

Abel's theorem has been further generalized by A. Hurwitz as follows [3] (see also [5]):

1.2 *Let V be a finite set, and $x = \{(v, x(v)) : v \in V\}$. For a set A, let $x(A) = \sum\{x(a) : a \in A\}$. Then*

$$(z+y)(z+y+x(V))^{|V|-1} = \sum\{z(z+x(A))^{|A|-1}) \cdot y(y+x(B))^{|B|-1}) : A \subseteq V, B = V \setminus A\}.$$

In this paper we describe a mechanism that provides infinitely many identities each being a generalization of Hurwitz's identity.

We use this mechanism to find some of such generalizations. As a byproduct we obtain combinatorial interpretations of all such identities. The "engine" of this mechanism is the relation between the so called forest volumes of graph–compositions and their bricks [4].

We will see that the volume formula from [4] applied to a very simple graph–composition gives a natural generalization of Hurwitz's identity. Namely this generalization corresponds to the graph–composition whose "frame" is the simple digraph with two vertices and one arc, and whose two bricks are a complete digraph and an empty graph (a graph with the empty edge set).

The mechanism we are going to describe provides for every acyclic digraph A a big variety of polynomial identities corresponding to a graph–composition whose frame is A.

The main notions and notation are given in Section 2. In Section 3 we describe this "engine". In Section 4 we give a simple generalization of Hurwitz's identity in order to elucidate the main idea of the general mechanism. and the main proof arguments. The main mechanism and its applications will be given in Section 5.

2 Main notions and notation

We consider directed graphs. All graph–theoretical notions that are used but not defined here can be found in [2].

A *directed graph* or simply a *digraph* G is a pair (V, E) where V is a finite non–empty set of elements (called *vertices* of G), $E \subseteq (V^2)$ where (V^2) is the set of ordered pairs of different elements of V (the elements of E are called *arcs* of G). Let $V(G) = V$ and $E(G) = E$.

A *source* (a *sink*) of a digraph G is a vertex v having no incoming (respectively, outgoing) edges in G. Let $L(G)$ and $R(G)$ denote the sets of sources and sinks of G, respectively.

A digraph is *acyclic* if it has no directed cycles.

A digraph F is a *subgraph* of G, written $F \subseteq G$ if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A digraph F of G is a *spanning subgraph* of G if $V(F) = V(G)$ and $E(F) \subseteq E(G)$.

Two spanning subgraphs F_1 and F_2 of G are *different* if $E(F_1) \neq E(F_2)$.

A *ditree* T is a digraph with the properties:
$(a1)$ T has no directed cycles, and
$(a2)$ for every vertex v in $V(T)$ except for one vertex, say r, there exists a unique arc $e_v = (v, t_v)$ starting at v.

The vertex r is called the *root of* A, and A is also called a *ditree rooted at* r. A *leaf* of a ditree A is a vertex having no incoming arc in A (or equivalently, a vertex of degree one in A). Clearly $R(A) = \{r\}$ and $L(A)$ is the set of leaves of A.

It is clear that a ditree is a digraph having exactly one component and no undirected cycles.

A *diforest* F is a digraph such that every it component is a ditree.

The *pointer* of a diforest A is a (the) function $f : V(A \backslash R(A)) \to V(A)$ such that $f(u) = v$ if $(u, v) \in E(A)$.

A *spanning ditree* (*spanning diforest*) of a digraph G is a spanning subgraph of G which is a ditree (respectively, a diforest).

Let $\mathcal{T}_r(G)$ denote the set of different spanning ditrees of G rooted at $r \in V(Q)$. Similarly let $\mathcal{F}(G)$ denote the set of different spanning diforests of G.

Let $x : V(G) \to K$ be a function where K is a commutative ring.

For a ditree T_r rooted at r, let $\mathcal{T}(T_r, x) = \prod\{x(v)^{d(v,T_r)-1} : v \in V(T_r)\}$.

Clearly $\mathcal{T}(T_r, x) = x_r^{d_{in}(r,T_r)-1} \prod\{x(v)^{d_{in}(v,T_r)} : v \in V(T_r) \setminus r\}$.

The *tree volume* (or $\mathcal{T}$*–volume*) of a digraph G *with respect to a given vertex* $r \in V(G)$ is

$$\mathcal{T}_r(G, x) = \sum\{\mathcal{T}(T, x) : T \in \mathcal{T}_r(G)\}.$$

Clearly $\mathcal{T}_r(G, x)$ is a polynomial in variables $x(v) : v \in V(G)$.

Let G^c be the digraph obtained from G by adding a new vertex c and the set of arcs $\{(v, c) : v \in V(G)\}$. For a function $x : V(G) \to K$, let $x^c : V(G^c) \to K$ be a function such that $x^c(v) = x(v)$ if $v \in V(G)$ and $x^c(c) = z \in K$, and so x is the restriction of x^c on $V(G)$.

The *forest volume* of G is

$$\mathcal{F}(z, G, x) = \mathcal{T}_c(G^c, x^c).$$

The forest volume of a weighted digraph G can also be viewed as a generating function of spanning diforests of G classified by their numbers of edges and degree sequences.

We recall that $[k, s] = \{k, \dots, s\}$ where k and s are integers and $k \le s$.

3 Forest polynomials of digraph compositions

Let G and $G_a = (B_a, g_a)$, $a \in A$, be disjoint weighted digraphs with $V(G) = A$, and $V(G_a) = B_a$, $a \in A\}$. Let $B = \cup\{B_a : a \in A\}$. Let $P_a = \{(a, b) : b \in B_a\}$ and $P = \cup\{P_a : a \in A$, and so $P = A \times B$.

The digraph Γ is called the G*–composition of* $\{G_a : a \in V(G)\}$, written $\Gamma = G\{G_a : a \in V(G)\}$, if $V = V(\Gamma) = P$ and for two vertices $v_1 = a_1b_1$ and $v_2 = a_2b_2$ of Γ, $(v_1, v_2) \in E(\Gamma)$ if and only if either $a_1 \ne a_2$ and $(a_1, a_2) \in E(G)$ or $a_1 = a_2 = a$ and $(b_1, b_2) \in E(G_a)$.

The graph G in $G\{G_a : a \in V(G)\}$ is called the *frame* and the graphs G_a, $a \in V(G)$, are called the *bricks* of the G–composition.

Let $x = \{(v, x(v) : v \in P$. Put

$x_a = x|_{L_a}$,

$x(G_a) = \sum\{x(a, b) : b \in B_a\}$,

$x(\Gamma) = \sum\{x(a, b) : a \in A, b \in B\}$,

$s : A \to K$ such that $s(a) = x(G_a)$ for $a \in A$, and

$d_a(G, s) = \sum\{s(c) : (a, c) \in E(G)\}$ for $a \in A$.

3.1 [4] $\mathcal{F}(z, G\{G_a : a \in V(G)\}, x) =$

$\mathcal{F}(z, G, s) \times \prod\{\mathcal{F}(z + d_a(G, s)), G_a, x_a) : a \in V(G)\}$.

We also need formulas for the forest volumes of some special digraphs.

3.2 [4] *Let K^0 and K^1 denote the empty and the complete digraphs with n vertices, respectively. Then $\mathcal{F}(z, K^p, x) = (z + px(K^p))^{n-1}$ where $p \in \{0, 1\}$.*

3.3 [4] *Let G be a digraph, $V(G) = \{v_1, \ldots, v_n\}$.*
Suppose that G is acyclic and $v_i \in L(G \setminus \{v_1, \ldots, v_{i-1}\})$, $i \in I_1^n$. Then

$$\mathcal{F}(z, G, x) = z^{-1} \prod\{z + d_u(G, x) : u \in V(G)\}.$$

4 A generalization of Hurwitz's identity

In order to illustrate the main idea of the mechanism of generating polynomial identities, we first describe and prove a simple generalization of Hurwitz's identity.

4.1 *Let U and V be disjoint non–empty sets, $x = \{(v, x(v)) : v \in V\}$ and $y = \{(u, y(u)) : u \in U\}$. Let $\mathcal{P}$ denote the set of all functions $Q : V \to U \cup c$ where $c \notin U$. For $P^{-1} \in \mathcal{P}$, let $P_a = \{v : Q(v) = a\}$.* [Clearly $\{P_a : a \in U \cup c\}$, where $c \notin U$, is a partition of V, i.e. $\cup\{P_a : a \in U \cup c\} = V$ and $P_a \cap P_b = \emptyset$ for $a \neq b$.] *Let V' be a digraph with the vertex set V. Then*

$(z + y(U))z^{|U|-1}\mathcal{F}(z + y(U)), V', x) =$

$\sum\{\mathcal{F}(z, P'_c, x) \prod\{zy(u)\mathcal{F}(y(u), P'_u, x) : u \in U\} : P^{-1} \in \mathcal{P}\}$

where P'_a is the subgraph of V' induced by P_a, $a \in U \cup c$.

Proof (uses **3.1**). Let U' be a digraph with the vertex set U and with the empty edge set, and G is a simple digraph with exactly two vertices u', v' and exactly one arc (v', u'). Let $\Gamma = G\{G_{u'}, G_{v'}\}$ where $G_{u'} = U'$ and $G_{v'} = V'$. Clearly $V(\Gamma) = U \cup V$. We will establish our identity by finding $\mathcal{F}(z, \Gamma, x \cup y)$ in two different ways.

(**p1**) We first find $\mathcal{F}(z, \Gamma, x \cup y)$, by using the volume formula for a G–composition.

By **3.1**,

$$\mathcal{F}(z, \Gamma, x \cup y) = \mathcal{F}(z, G, s)\mathcal{F}(z + d_{u'}(G, s)), U', y)\mathcal{F}(z + d_{v'}(G, s)), V', x) \tag{4.1}$$

where $s = \{(u', s(u'), (v', s(v')), s(u') = y(U)$, and $s(v') = x(V)$.
Since G has two vertices u', v' and exactly one arc (v', u'), we have:

$\mathcal{F}(z, G, s) = z + y(U)$ and $d_{v'}(G, s) = y(U)$. Therefore

$\mathcal{F}(z + d_{v'}(G, s)), V', x) = \mathcal{F}(z + y(U)), V', x)$

Since U' has no arcs, $\mathcal{F}(z + d_{u'}(G, s)), U', y) = z^{|U|-1}$. Thus from (4.1) we have:

$$\mathcal{F}(z, \Gamma, x \cup y) = (z + y(U))z^{|U|-1}\mathcal{F}(z + y(U)), V', x). \tag{4.2}$$

(**p2**) Now let us find $\mathcal{F}(z, \Gamma, x \cup y)$ in another way. Note that Γ is a simple digraph. Let $x \cup y = h$ and $h^c = h \cup (c, z)$. If $H \subseteq G$ and $f : V(G) \to K_v)$, we will write $\mathcal{F}(z, H, f)$ and $T(H, f$ instead of $\mathcal{F}(z, H, f|_{V(H)})$ and $T(H, f|_{V(H)}$.

By the definition of the forest volume of a digraph,

$$\mathcal{F}(z, \Gamma, h) = \mathcal{T}_c(\Gamma^c, h^c) = \sum\{\mathcal{T}_c(T, h^c) : T \in \mathcal{T}_c(\Gamma^c)\} \tag{4.3}$$

where for $T \in \mathcal{T}_c(\Gamma^c)$ we have: $\mathcal{T}_c(T, h^c) = \prod\{h(v)^{d(v,T)-1} : v \in V(T)\}$.

Since Γ has no arc (u, v) with $u \in U$ and $v \in V$, every spanning ditree of Γ^c contains the arc set $(U, c) = \{((u, c) : u \in U\}$.

For a spanning ditree T of Γ^c, let $T' = T \setminus (U, c)$. Clearly T' is a spanning diforest of Γ^c, and so each component of T' is a diforest. Let C_u be the component of T' containing $u \in U$. Since there is no arc going out of u in Γ, clearly u is a root of C_u. Let $\mathcal{C}_U(T') = \{C_u : u \in U\}$ and (T_c) denote the components of T' containing c.

$$\mathcal{T}_c(T, h^c) = \mathcal{T}_c(T_c, h^c) \prod\{zy(u)\mathcal{T}_u(C, h) : C \in \mathcal{C}_U(T')\}. \tag{4.4}$$

By (4.3),

$$\mathcal{F}(z, \Gamma, h) = \sum\{\mathcal{T}_c(T, h^c) : T \in \mathcal{T}_c(\Gamma^c)\} = \sum\{S(\Gamma, h^c, Q) : Q \in \mathcal{P}\} \tag{4.5}$$

where $S(\Gamma, h^c, P^{-1}) = \sum\{\mathcal{T}_c(T, h^c) : T \in \mathcal{T}_c(\Gamma^c), V(C \setminus u) = P_u, C_u \in \mathcal{C}_U(T'), u \in U\}$.

By (4.4), $S(\Gamma, h^c, P^{-1}) = \sum\{\mathcal{T}_c(T^c, h^c) \prod\{zy(u)\mathcal{T}_u(C, h) :$

$C \in \mathcal{C}_U(T')\} : T \in \mathcal{T}_c(\Gamma^c), V(C \setminus u) = P_u, C_u \in \mathcal{C}_U(T'), u \in U\}$.

For $a \in U \cup \{c\}$, let P'_a and $\acute{P}_a$ denote the subgraphs of Γ^c induced by V_a and $V_a \cup a$, respectively. Then

$S(\Gamma, h^c, P^{-1}) = (\sum\{\mathcal{T}_c(T, h^c) : T \in \mathcal{T}_c(\acute{P}_c)\})$

$\prod\{zy(u) \sum\{\mathcal{T}_u(T, h) : T \in \mathcal{T}_u(\acute{P}_u)\} : u \in U\} =$

$\mathcal{F}(z, P'_c, x) \prod\{zy(u)\mathcal{F}(y(u), P'_u, x) : u \in U\}$.

Since U' is a complete digraph, clearly P'_c and P'_u are also complete digraphs. Therefore

$\mathcal{F}(z, V'_c, x) = (z + x(P_c))^{|P_c|-1}$ and $\mathcal{F}(y(u), V'_u, x) = (y(u) + x(P_c))^{|P_u|-1}$.

Now the required identity follows from (4.3) and the last equation. □

From **4.1** we have in particular:

4.2 *Let U and V be disjoint sets, $x = \{(v, x(v)) : v \in V\}$ and $y = \{(u, y(u)) : u \in U\}$. Then*

$(z + y(U))z^{|U|-1}(z + y(U) + x(V))^{|V|-1} =$

$\sum\{((z + x(P_c))^{|P_c|-1} \prod\{zy(u)(y(u) + x(P_u))^{|P_u|-1}) : u \in U\} : P^{-1} \in \mathcal{P}\}$.

Proof (uses **3.2** and **4.1**). Let in **4.1** V' be a complete digraph. Let as in **4.1**, P'_a denote the subgraphs of V' induced by P_a, $a \in U \cup c$. Since U' is a complete digraph, clearly P'_c and P'_u are also complete digraphs. Therefore by **3.2**,

$$\mathcal{F}(z, V', x) = (z + x(V))^{|V|-1} \text{ and } \mathcal{F}(z, P'_a, x) = (z + x(P_a))^{|P_a|-1} \text{ for } a \in U \cup c.$$

Now the requied identity follows from **4.1**. □

Hurwitz's identity **1.2** is a particular case of **4.2** when $|U| = 1$.

5 Generating polynomial identities

Let G be a digraph with $V(G) = A$ and $N_a = \{b \in A : (a, b) \in E(G)\}$. Let V_a, $a \in A$, be a collection of disjoint non-empty sets, $V = \cup\{V_a : a \in A\}$, and $c \notin V$. Let $L = L(G)$ and $R = R(G)$ denote the set of sources and sinks of G, respectively. Let $V^a = \cup\{V_b : b \in N_a\}$ for $a \in A \setminus R$. Let $\mathcal{P}_a$ denote the set of functions $Q : V_a \to V^a \cup c$, $a \in A$. For $P_a^{-1} \in \mathcal{P}$, let $P_a(u) = \{v \in V_a : P_a^{-1}(v) = u\}$. Clearly $\{P_a(u) : u \in V^a \cup c\}$ is a partition of V_a, i.e. $\cup\{P_a(u) : u \in V^a \cup c\} = V_a$ and $P_a(u) \cap P_a(w) = \emptyset$ for $u \neq w$.

By using **3.1** and **3.3**, we obtain:

5.1 *Let G above be an ayclic digraph with $V(G) = A$, and let $x = \{(v, x(v)) : v \in V \cup c\}$, and $s = \{(a, s(a)) : a \in A\}$ where $s(a) = x(V_a)$. Let G_a be a digraph with $V(G_a) = V_a$ for every $a \in L(G)$. Then*

$$\mathcal{F}(x(c), G, s) \times \prod\{\mathcal{F}(x(c) + x(V^a)), G_a, x) : a \in L\} \times$$
$$\prod\{(x(c) + x(V^a)))^{|V_a|-1} : a \in A \setminus L\} =$$
$$((x(c))^{-1} \sum\{\prod\{x(c)\mathcal{F}(x(c), P'_a(c), x) : a \in L\} \times$$
$$\prod\{x(u)\mathcal{F}(x(u), P'_a(u), x) : u \in V^a, a \in L\} \times$$
$$\prod\{(x(u))^{|P_a(u)|} : u \in V^a, a \in A \setminus L\} : P_a^{-1} \in \mathcal{P}_a, a \in A \setminus R\}$$

Here $P'_a(u)$ is the subgraph of G_a induced by $P_a(u)$ for $u \in V^a$ and $a \in A \setminus R$, and $\mathcal{F}(z, G, x)$ is given by **3.3**.

The proof of this identity uses the arguments similar to that in the proof of **4.1** where the digraph G (of two vertices and one arc) is replaced by an arbitrary acyclic digraph.

Theorem **5.1** provides a mechanism of generating polynomial identities. We can obtain a big variety of identities by considering in **5.1** various specific digraphs G and G_a's. For example, we can obtain specific identities by replacing G by a ditree T_r (see the next theorem). We also can replace the G_a's by complete or empty digraphs, and use formula **3.2**.

The identity **5.1** includes the forest volumes of the G_a's and their induced subgraphs. Therefore it is also natural to consider instead of the class $\mathcal{K}$ of complete digraphs for the G_a's (as suggested above) another class $\mathcal{G}$ that is closed under the digraph operation of taking an induced subgraph. For example, natural classes of digraphs to consider as $\mathcal{G}$ are the classes of bipartite, multipartite, threshold, and more generally, totally decomposable digraphs [2, 4].

From **3.3** and **5.1** we have in particular:

5.2 *Let T_r be a ditree with $V(T_r) = A$ rooted at $r \in A$, and let $f : A \setminus r \to A$ be the pointer of T_r. Let V_a, $a \in A$, be a collection of disjoint non–empty sets, and $n_a = |V_a|$, $V = \cup\{V_a : a \in A\}$, $x = \{(v, x(v)) : v \in V \cup c\}$, and $s = \{(a, s(a)) : a \in A\}$ where $s(a) = x(V_a)$. Let $\mathcal{P}_a$ denote the set of all functions $Q : V_a \to V_{f(a)} \cup c$ where $c \notin V_{f(a)}$ and $a = A \setminus r$. For $P_a^{-1} \in \mathcal{P}$, let $P_a(u) = \{(v \in V_a : P^{-1}(v) = u\}$.* [Clearly $\{P_a(u) : u \in V_{f(a)} \cup c\}$ is a partition of V_a, i.e. $\cup\{P_a(u) : u \in V_{f(a)} \cup c\} = V_a$ and $P_a(u) \cap P_a(w) = \emptyset$ for $u \neq w$.] *Let G_a be a digraph with $V(G_a) = V_a$, $a \in L(T_r)$. Then*

$$\prod\{(x(c) + s(a) : (v, a) \in E(T_r)\} \times \prod\{\mathcal{F}(x(c) + s(f(a)), G_a, x) : a \in L\} \times$$

$$\prod\{(x(c) + s(f(a)))^{|V_a|-1} : a \in A \setminus L\} =$$

$$((x(c)){-}1 \sum\{\prod\{x(c)\mathcal{F}(x(c), P'_a(c), x) : a \in L\} \times$$

$$\prod\{x(u)\mathcal{F}(x(u), P'_a(u), x) : u \in V_{f(a)}, a \in L\} \times$$

$$\prod\{(x(u))^{|P_a(u)|} : u \in V_{f(a)}, a \in A \setminus L\} : P_a^{-1}, a \in A \setminus r\}$$

where $P'_a(u)$ is the subgraph of G_a induced by $P_a(u)$ for $u \in V_{f(a)}$ and $a \in A \setminus r$.

A particular case of this identity is when

$$\mathcal{F}(x(c) + s(f(a)), G_a, x) = (x(c) + s(a) + s(f(a))^{|V_a|_a - 1},$$

$$\mathcal{F}(x(c), P'_a(c), x) = (x(c) + x(P_a(c))^{|P_a(c)|-1}, \text{ and}$$

$$\mathcal{F}(x(u), P'_a(u), x) = (x(u) + x(P_a(u))^{|P_a(u)|-1}.$$

Clearly **4.1** and **4.2** are particular cases of **5.2** when T_r has exactly one arc.

References

[1] N. Abel, A generalization of the binomial formula, *Journal für die reine und angewandle Mathematik* **1** (1826) 159–160.

[2] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, North Holland, Amsterdam, 1976.

[3] A. Hurwitz, Uber Abel's Verallgemeinerung der binomischen Formel, *Acta Mathematica* **26** (1902) 199–203.

[4] Kelmans A.K., Pak I., Postnikov A. Tree and forest polynomials of graphs, *RUTCOR Research Report 47-99*, Rutgers University (1999).

[5] D.E. Knuth, *The art of computer programming*, **1** Addison Wesley, 1973.

[6] J. Riordan, *Polynomial identities*, New York, Wiley, 1968.