Chern Forms on Flag Manifolds and Forests  
(Extended Abstract)

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Summary

Let $\mathcal{A}_n$ be the ring generated by the Chern 2-forms of $n$ standard hermitian line bundles over the flag manifold $\text{SL}(n, \mathbb{C})/B$. We prove a conjecture from [3] that the dimension of $\mathcal{A}_n$ is equal to the number of forests on $n$ labelled vertices. We present an explicit construction for a monomial basis in $\mathcal{A}_n$. More generally, the results naturally extend to a wider class of rings, whose bases are labelled by generalized parking functions.
1 Main Results

Let $Fl_n = \text{SL}(n, \mathbb{C})/B$ be the manifold of complete flags in $\mathbb{C}^n$. The manifold $Fl_n$ comes equipped with a flag of of tautological vector bundles $E_0 \subset E_1 \subset \cdots \subset E_n$ and associated sequence of line bundles $L_i = E_i/E_{i-1}, i = 1, \ldots, n$. The $L_i$ possess natural hermitian structures induced from the standard hermitian metric $\sum z_i \overline{z}_i$ on $\mathbb{C}^n$. For $i = 1, \ldots, n$, we denote by $w_i$ the 2-dimensional Chern form (or curvature form) on $Fl_n$ of the hermitian line bundle $L_i$. The $w_i$ are also called the curvature forms. They represent the Chern classes $c_1(L_i)$ in the 2-dimensional cohomology of $Fl_n$. The forms $w_i$ are invariant under the action of the unitary group $U_n$ on $Fl_n$.

The main purpose of the present paper is to investigate the ring $A_n$ generated by the forms $w_1, \ldots, w_n$. As an additive group, $A_n$ is a free abelian group. The ring $A_n$ is graded: $A_n = A_n^0 \oplus A_n^1 \oplus A_n^2 \cdots$. The component $A_n^k$ consists of $2k$-dimensional forms. The cohomology ring $H^*(Fl_n, \mathbb{Z})$ of the flag manifold is a quotient of the ring $A_n$, since the former is generated by the Chern classes $c_1(L_i)$.

Recall that a forest is a graph without cycles. For a forest $F$ on vertices labelled $1, \ldots, n$, let us construct a tree $T$ by adding a new vertex (root) connected with the maximal vertices in the connected components of $F$. An inversion in $F$ is a pair $1 \leq i < j \leq n$ such that the vertex labelled $j$ lies on the shortest path in $T$ from the vertex labelled $i$ to the root.

Our primary result is the following statement. Its first part was initially conjectured in [3] and the second part was then guessed by R. Stanley.

**Theorem 1** The dimension of the ring $A_n$ is equal to the number of forests on $n$ labelled vertices. Moreover, the dimension of a graded component $A_n^k$ is equal to the number of forests on $n$ labelled vertices with exactly $\binom{n}{2} - k$ inversions.

To formulate our further results, we need some extra notation. We say that a sequence $a = (a_1, a_2, \ldots, a_n)$ of nonnegative integers is a strictly parking function if it satisfies the following two conditions:

1. For $r = 0, 1, \ldots, n$, we have $\#\{i \mid a_i \geq n - r\} \leq r$.
2. For $r \in \{1, \ldots, n\}$ such that $\#\{i \mid a_i \geq n - r\} = r$, let $j = \min\{i \mid a_i \geq n - r\}$. Then $a_j = n - r$.

Let $P_n$ be the set of all strictly parking functions. We remark that a sequence $a$ that satisfies the first of these two conditions is usually called a parking function.

For a sequence $a = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n$, we denote $w^a = w_1^{a_1} w_2^{a_2} \cdots w_n^{a_n}$.

**Theorem 2** The monomial forms $w^a, a \in P_n$, form a linear basis in the ring $A_n$.

Theorem 2 together with the following combinatorial statement imply Theorem 1. Let $|a| = a_1 + a_2 + \ldots + a_n$. 

2
Theorem 3  The number of elements \( a \in P_n \) such that \(|a| = k\) is equal to the number of forests on \( n \) labelled vertices with exactly \( \binom{n}{2} - k \) inversions.

The proof of this theorem is based on an explicit bijection between forests and strictly parking functions.

A description of the ring \( A_n \) in terms of generators and relations was given in [3]. Let \( I_n \) be the ideal in the polynomial ring \( \mathbb{Z}[x_1, \ldots, x_n] \) generated by \( 2^n - 1 \) polynomials of the form

\[
(x_{i_1} + \cdots + x_{i_r})^{r(n-r)+1},
\]

where \( \{i_1, \ldots, i_r\} \) is any nonempty subset of \( \{1, \ldots, n\} \).

Theorem 4  [3]  The ring \( A_n \) is canonically isomorphic, as a graded ring, to the quotient \( \mathbb{Z}[x_1, \ldots, x_n]/I_n \). The isomorphism is given by sending the generators \( w_i \) of \( A_n \) to the corresponding \( x_i \).

Let us also define the ideal \( J_n \) generated by \( 2^n - 1 \) monomials of the form

\[
(x_{i_1} \cdots x_{i_r})^{n-r} x_{i_1},
\]

where \( 1 \leq i_1 < \cdots < i_r \leq n \) is any nonempty subset of \( \{1, \ldots, n\} \). Finally, let \( B_n = \mathbb{Z}[x_1, \ldots, x_n]/J_n \).

Lemma 5  The monomials \( x^a, a \in P_n \), form a linear basis of the ring \( B_n \).

Our proof of Theorem 2 is based on a construction of a sequence of rings that interpolates between \( A_n \) and \( B_n \). We then prove by induction an analogous statement for all rings in this sequence. Lemma 5 provides the base of the induction. In particular, we deduce the following statement.

Corollary 6  The rings \( A_n \) and \( B_n \) have the same Hilbert series.

Let us also consider a more general ring \( A_{nk} \), \( 1 \leq k \leq n \), generated by the first \( k \) Chern forms \( w_1, w_2, \ldots, w_k \) (see [3]). It is not hard to show that \( A_{nk} \) is isomorphic to the ring generated by any \( k \)-tuple of the Chern forms \( w_j, \ldots, w_{jk} \). It is clear that \( A_{nn} = A_n \) and \( A_{nn-1} = A_n \). (The latter is due to the identity \( w_1 + \cdots + w_n = 0 \).)

Theorem 7  The dimension of the ring \( A_{nk} \) is equal to the number of forests on \( 2n - k \) labelled vertices such that the first \( n - k \) vertices belong to \( n - k \) different connected components.

Let \( f_n(q) \) be the generating function \( f_n(q) = \sum_F q^{d(F)} \), where the sum is over all forests \( F \) on \( n + 1 \) labelled vertices and \( d(F) \) is the degree of the first vertex, i.e. the number of edges that emanate from it. The previous theorem is equivalent to the following statement.
Corollary 8 The dimension of $A_{nk}$ is equal to $f_n(n-k)$.

The ring $A_{nk}$ is canonically isomorphic to the quotient of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ modulo the ideal generated by $2^k - 1$ polynomials of the form (1), where $i_1, \ldots, i_r \leq k$.

Analogously, let $B_{nk}$ be the quotient of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ modulo the ideal generated by $2^k - 1$ polynomials of the form (2), where $i_1, \ldots, i_r \leq k$. Clearly, both $A_{nk}$ and $B_{nk}$ are graded rings. Corollary 6 can be generalized as follows.

Theorem 9 The rings $A_{nk}$ and $B_{nk}$ have the same Hilbert series.

2 Remarks and Open Problems

A natural open problem is to extend the results to a partial flag manifold $SL_n/P$, where $P$ is a parabolic subgroup.

Let $E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_r = \mathbb{C}^n$, $\dim E_k = i_1 + i_2 + \ldots + i_k$, be the tautological bundles over $SL_n/P$ and denote by $L_k(P)$ the quotients $E_k/E_{k-1}$, $k = 1, \ldots, r$. The standard hermitian form in $\mathbb{C}^n$ induces a hermitian structure on each $L_k(P)$ and gives rise to the Chern forms $w_1^k, w_2^k, \ldots, w_r^k$ of dimensions $2, 4, \ldots, 2i_k$. Explicit formulas for these forms can be found in [1].

Let $A_n(P)$ be the ring generated by the forms $w_k^j$. It is a natural extension of the ring $A_n$. The cohomology ring $H^*(SL_n/P, \mathbb{Z})$ is a quotient of the ring $A_n(P)$.

Problem 10 Investigate the ring $A_n(P)$. Find a description for this ring in terms of generators and relations. Find the dimension of the ring $A_n(P)$ and its Hilbert series.

Remark 11 In the case of the Grassmannian $G_{n,k}$ (i.e. when $P$ is a maximal parabolic subgroup) the ring $A_n(P)$ coincides with the cohomology ring $H^*(G_{n,k})$, cf. [1]. In particular, its dimension is equal to $\binom{n}{k}$.

The ring $A_n$ is related to the ring of all $U_n$-invariant forms, which recently appeared in [4, 5]. The latter ring has an additive basis that consists of Eulerian digraphs on $n$ labelled vertices.

Problem 12 Find an explicit description in terms of generators and relations for the ring of all $U_n$-invariant forms.

There is an analogy between the cohomology ring $H^*(Fl_n, \mathbb{Z})$ of the flag manifold and the Orlik-Solomon algebra $OS_n$ of the braid hyperplane arrangement, which
consists of all hyperplanes \( \{ x_i = x_j \} \), \( 1 \leq i < j \leq n \). For example, the dimensions of these two algebras are equal to each other.

Let \( \widetilde{OS}_n \) be the Orlik-Solomon algebra of a generic affine deformation of the braid arrangement, which consists of the hyperplanes \( \{ x_i = x_j + \epsilon_{ij} \} \), where \( \epsilon_{ij} \) are generic real numbers. The analogy between \( H^*(F_{\ell}, \mathbb{Z}) \) and \( OS_n \) seems to extend to the ring \( \mathcal{A}_n \) on one side and \( \widetilde{OS}_n \) on the other side. For example, the dimension of \( \widetilde{OS}_n \) is equal to the number of forests on \( n \) labelled vertices, see [2]. It would be interesting to clarify and study this relationship.

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References


