ENUMERATION OF TREES AND ONE AMAZING REPRESENTATION OF THE SYMMETRIC GROUP

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Abstract. In this paper we present a systematic approach to enumeration of different classes of trees and their generalizations. The principal idea is finding a bijection between these trees and some classes of Young diagrams or Young tableaux. The latter arise from the remarkable representation of the symmetric group studied by Haiman in connection with diagonal harmonics (see [7]).

Define the vector space $V \simeq \langle x_{\sigma(1)}^{a_1} \dots x_{\sigma(n)}^{a_n} | \sigma \in S_n, 0 \le a_i \le i-1, 1 \le i \le n \rangle$. Let the symmetric group S_n act on V_n by the permutation of variables. It is known that $dim(V_n) = (n+1)^{n-1}$ is equal to the number of labeled trees, and $dim(V_n)^{S_n} = \frac{1}{n+1}\binom{2n}{n}$ is equal to the number of plane trees with n vertices. There are combinatorial interpretations for the other multiplicities. We generalize all the results in case of k-dimensional trees and $(k+1)$-ary trees.

1. Introduction.

Define a vector space

$$V \simeq \langle x_{\sigma(1)}^{a_1} \dots x_{\sigma(n)}^{a_n} |\ \sigma \in S_n,\ 0 \le a_i \le i-1,\ 1 \le i \le n \rangle \tag{1–1}$$

Let the symmetric group S_n act on V by the permutation of variables. Call this representation T_n. It has been studied by Mark Haiman in [7] in connection with diagonal harmonics. He also found the following property of T_n.

Take a vector space $U \simeq \mathbb{C}^{n+1}$ and $W \simeq U^{\otimes n}$. Let the symmetric group S_n act on V by permutation of coordinates. Then in a ring of characters

$$T_n = \frac{1}{n+1} W \tag{1–2}$$

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From here we immediately have:

$$dim(T_n) = (n+1)^{n-1} \tag{1–3}$$

i.e. $dim(T_n)$ is equal to the number of labeled trees with $n+1$ vertices (see e.g. [6,8,17,23]).

Denote S^λ an irreducible representation of S_n associated with the Young diagram λ (see [9,16,22,24]). Now we can find multiplicities of S^λ in T_n.

Consider an action of the $GL(n+1)$ on W by linear transformations on each V in $W \simeq V \otimes V \otimes \cdots \otimes V$. By the Schur–Weyl duality (see [26]) and the hook-content formula for the dimension of the irreducible representation of $GL(n+1)$ (see e.g. [9,16,24]) , the multiplicity of S^λ in S_n-module W is given by the following formula:

$$c(T_n, S^\lambda) = \frac{1}{n+1} \prod_{(i,j)\in\lambda} \frac{n-i+j+1}{h(i,j)} \tag{1–4}$$

where $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length.

In particular, we have:

$$dim(T_n)^{S_n} = \frac{1}{n+1}\binom{2n}{n} \tag{1–5}$$

i.e. the dimension of invariants is equal to the Catalan number (see [7]). This number has numerous interpretations, such as the number of plane trees with $n+1$ vertices or to the number of triangulations of a polygon with $n+2$ vertices (see e.g. [6,8,15,23]).

More generally, we have:

$$c(T_n, S^{(n-l,1^l)}) = \frac{1}{n+1}\binom{2n-l}{n}\binom{n-1}{l} \tag{1–6}$$

i.e. this multiplicity equal to the number of polygonal subdivisions of $(n+2)$-gon with $(n-l)$ regions (see [20]).

The main goal of our paper is to explain that all the previous observations are not coincedental, but a part of the general picture. We shall present a set of bijections which prove all these and many other results with their k-dimensional generalizations.

In Section 2 of the abstract we shall state precisely what kind of combinatorial object appear in this case. In Section 3 we point at the k-dimensional generalizations.

2. Main results

We shall recall definitions of Young diagrams and Young tableaux (see e.g [16,22,24]).

The *Young diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a set of 1×1 lattice squares with centers at points (i,j), $1 \le j \le \lambda_i$ (see diagram $(4,3,1)$ on Fig. 3–1. Skew Young diagram is a set theoretic difference of two Young diagrams. *Horisontal* *stripe* is a skew Young diagram which contains exactly one square in each column. C_n*-diagram* is a horisontal stripe with n squares inside the staircase shape diagram $\delta_n = (n, n-1, \dots, 1)$. Denote $\mathcal{CD}_n$ the set of all C_n-diagrams.

Young tableau of a shape $\lambda \vdash n$ is a function on the Young diagram λ into $\mathbb{Z}$, which is nondecreasing in rows and strictly increasing in columns. Young tableau is *standard* if it's a bijection into $\{1, 2, \dots\}$. C_n-tableau is a standard Young tableau of shape C_n-diagram. We draw them by putting values inside the squares. Denote $\mathcal{CT}_n(\lambda)$ the set of C_n-tableaux of shape λ, and $\mathcal{CT}_n$ the set of all C_n-tableaux.

We call $\nu_1 \circ \nu_2$ the disjoint union of diagrams ν_1 and ν_2. Then horisontal stripe is simply a union of rows.

The *weight* of a tableau A is a sequence $\omega(A) = (w_1, w_2, \dots)$, where ω_i is the number of elements i in A. Denote $\mathcal{CT}_n(\lambda)$ the set of all CT_n-tableaux A with weight $\omega(A) = \lambda$.

With each skew Young diagram ν is associated a representation S^ν of S_n (see e.g. [16,22,26]). In particular, monomial representation M^λ correspons to a skew diagram $\tilde{\lambda} = \lambda_1 \circ \lambda_2 \circ \dots$.

Proposition 2.1.

$$T_n = \sum_{\nu \in \mathcal{CD}_n} S^\nu \tag{2–1}$$

From here and the Young rule (see e.g. [M, St], we have:

Proposition 2.2.

$$c(T_n, S^\lambda) = |\mathcal{CT}_n(\lambda)| \tag{2–2}$$

Definition 2.3. *A sequence $a = (a_1, \dots, a_n)$, $1 \le a_i \le n$ is called majorating sequence if $\#\{j | a_j \le i\} \ge i$ for all $i = 1, \dots, n$ (see [13,14]). Denote $\mathcal{M}_n$ a set of all majorating sequences.*

Define a Dyck sequence $b = (b_1, \dots, b_{2n})$, $b_i \in \{1; -1\}$ by the following inequalities: $b_1 + b_2 + \dots + b_i \ge 0$, $i = 1, \dots, 2n-1$, and $b_1 + \dots + b_{2n} = 0$. Denote $\mathcal{DS}_n$ the set of all Dyck sequences of length $2n$.

Definition 2.4. *Define the following sets of trees:*

1) $\mathcal{L}_n$ be a set of labeled trees with n vertices,

2) $\mathcal{P}_n$ be a set of plane trees with n vertices ,

3) $\mathcal{B}_n$ be a set of nonlabeled binary trees with n vertices,

4) $\mathcal{IB}_n$ be a set of increasing in both directions binary trees with n vertices,

5) $\mathcal{RB}_n$ be the set of increasing to the right binary trees with n vertices,

6) $\mathcal{IL}_n$ be the set of increasing labeled trees with n vertices.

Define surjections $\rho : \mathcal{L}_n \to \mathcal{P}_n$ and $\tau : \mathcal{RB}_n \to \mathcal{B}_n$ by forgetting labels of the vertices. Analogously $\epsilon : \mathcal{CT}_n \to \mathcal{CD}_n$ is a surjection which maps tableau into it's shape.

Main Theorem 2.5. *There is a set of natural bijections which form the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{DS}_n \arrow[d, "\eta^*"'] & \mathcal{M}_n \arrow[l, "\kappa"] \arrow[d, "\eta"'] & \\
\mathcal{CD}_n \arrow[d, "\psi^*"'] & \mathcal{CT}_n \arrow[l, "\epsilon"] \arrow[d, "\psi"'] & \\
\mathcal{B}_n \arrow[d, "\varphi^*"'] & \mathcal{RB}_n \arrow[l, "\tau"] \arrow[d, "\varphi"'] & \mathcal{IB}_n \arrow[l, "\iota_{\mathcal{B}}"] \arrow[d, "\varphi"'] \\
\mathcal{P}_n & \mathcal{L}_n \arrow[l, "\rho"] & \mathcal{IL}_n \arrow[l, "\iota_{\mathcal{L}}"]
\end{tikzcd}$$

where ι is simply an inclusion operator.

Remark. *We actually present in our paper all these bijections. Some of them are new, some can be found in the literature (see* [1,4,5,10,12,23,25]*)*

4. k-GENERALIZATIONS

Here we point on a k-generalization of all results in Section 2.

We can analogously define generalized vector space V as follows:

$$V \simeq \langle x^{a_1}_{si(1)} \dots x^{a_n}_{si(n)} | \, \sigma \in S_n, \ 0 \le a_i \le k(i-1), \ 1 \le i \le n \rangle \tag{3–1}$$

with action of the symmetric group S_n as before. Call this representation T^k_n. It has a dimension

$$dim(T^k_n) = (kn+1)^{(n-1)} \tag{3–2}$$

i.e. equal to the number of k-dimensional trees (see [2,8]).

Using Schur-Weyl duality and the hook-length formula we have

$$c(T^k_n, S^\lambda) = \frac{1}{kn+1} \prod_{(i,j) \in \lambda} \frac{kn+i-j+1}{h(i,j)} \tag{3–3}$$

In particular,

$$dim(T^k_n)^{S_n} = \frac{1}{kn+1} \binom{(k+1)n}{n} \tag{3–4}$$

i.e. the dimension of invariants is equal to the k-Catalan number (see f.e. [3]) and the number of $(k+1)$-ary trees (see [11]).

We claim that in this situation everything works the same way, and we get an exact analog of the Main Theorem.

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