Problem 1. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a Young diagram that fits inside the $k \times n$ rectangle. Consider the subset $S_\lambda$ of the Grassmannian $Gr(k,n)$ over a finite field $\mathbb{F}_q$ that consists of the elements that can be represented by $k \times n$ matrices $A$ with 0’s outside the shape $\lambda$. For example, for $n = 4$ and $k = 2$, $S_{(4,1)}$ is the subset of elements of $Gr(2,4)$ representable by matrices of the form

$$\begin{pmatrix}
* & * & * & * \\
* & 0 & 0 & 0
\end{pmatrix}$$

In parts 1, 2, 3 assume $n = 2k$ and $\lambda = (2k, 2k-2, 2k-4, \ldots, 2)$.

1. Find a combinatorial expression for the number of elements of $S_{(2k,2k-2,\ldots,2)}$ (over $\mathbb{F}_q$). Show that it is a polynomial in $q$.

2. Let $f_k(q)$ be the polynomial from part 1. Calculate $f_k(1)$, $f_k(0)$, and $f_k(-1)$.

3. Let $g_k(q) = q^d f_k(q^{-1})$, where $d$ is the degree of the polynomial $f_k(q)$. Find the maximal power of 2 that divides the number $g_k(5)$.

4. Generalize (some of) the above to other Young diagrams $\lambda$.

5. What about skew shapes $\lambda/\mu$?

Problem 2. Let $C[\Delta_I]$ be the polynomial ring in $\binom{n}{k}$ (independent) variables $\Delta_I$, $I \in \binom{[n]}{k}$. Let $I_{kn} = \langle \Delta_{i_1 \ldots i_k} \Delta_{j_1 \ldots j_k} - \sum \Delta_{i_1' \ldots i_k'} \Delta_{j_1' \ldots j_k'} \rangle$ be the ideal in $C[\Delta_I]$ whose generators correspond to the Plücker relations (for all $r$).

Show that $I_{kn}$ is a prime ideal. Deduce that $I_{kn}$ consists of all polynomials in $C[\Delta_I]$ that vanish on the image of the Grassmannian $Gr(k,n,\mathbb{C})$ in the projective space $\mathbb{CP}^{n\choose k-1}$ under the Plücker embedding.

Problem 3. Let $M \subseteq \binom{[n]}{k}$. Is it true that the following three properties are equivalent?

- Exchange Property: For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that $(I \setminus \{i\}) \cup \{j\} \in M$.

- Stronger Exchange Property: For any $I, J \in M$ and any $i \in I$, there exists $j \in J$ such that both $(I \setminus \{i\}) \cup \{j\}$ and $(J \setminus \{j\}) \cup \{i\}$ are in $M$.

- Even Stronger Exchange Property: For any $I, J \in M$, any $r \geq 1$, and any $i_1, \ldots, i_r \in I$, there exist $j_1, \ldots, j_r \in J$ such that both $(I \setminus \{i_1, \ldots, i_r\}) \cup \{j_1, \ldots, j_r\}$ and $(J \setminus \{j_1, \ldots, j_r\}) \cup \{i_1, \ldots, i_r\}$ are in $M$.

Prove the equivalence of (some of) these properties or construct counterexamples.
**Problem 4.** Check that the Fano plane satisfies the exchange axiom and show that this matroid is not realizable over $\mathbb{R}$.

**Problem 5.** Prove the equivalence of the 3 definitions of matroids: the definition in terms of exchange axiom, the definition in terms of Gale minimal elements, and the definition in terms of matroid polytopes.

**Problem 6.**
1. Prove that image of the Grassmannian $Gr(k, n, \mathbb{C})$ under the moment map is a convex polytope.
2. Describe the moment map image of (the closure of) the Schubert cell $\Omega_{(2,1)} \subset Gr(2, 4, \mathbb{C})$.
3. Calculate the normalized volume of the moment map image of $\overline{\Omega}_\lambda \subset Gr(k, n, \mathbb{C})$ for any $\lambda$.

**Problem 7.**
1. Prove that the $(n - 1)$ dimensional volume of the permutohedron $P_n = \text{ConvexHull}\{(w_1, \ldots, w_n) \mid w \in S_n\}$ is $n^{n-2}$.

**Problem 8.** Find an expression for the Ehrhart polynomial $i(P, t) := \#(tP \cap \mathbb{Z}^n)$, $t \in \mathbb{Z}_{\geq 0}$, of the hypersimplex $P = \Delta_{kn}$ using inclusion-exclusion.

**Problem 9.**
1. Prove geometrically Pieri’s formula for the Schubert classes by intersecting Schubert varieties.
2. Prove geometrically the duality formula by intersecting Schubert varieties.

**Problem 10.** Show that the ideal in the ring of symmetric polynomials $\Lambda_k := \mathbb{C}[x_1, \ldots, x_k]^{S_k}$ generated by all Schur polynomials $s_\lambda(x_1, \ldots, x_k)$ for shapes $\lambda$ that don’t fit inside the $k \times (n - k)$-rectangle coincides with the ideal

$$I = \left\langle h_{n-k+1}, h_{n-k+2}, \ldots, h_n \right\rangle,$$

where $h_i = h_i(x_1, \ldots, x_k)$ are the complete homogeneous symmetric polynomials.

**Problem 11.** Prove the equivalence of the following versions of the Littlewood-Richardson rules for $c^\nu_{\lambda\mu}$: the classical LR-rule, the honeycomb version of LR-rule, the web diagram version of LR-rule.

**Problem 12.** Explicitly show (without using LR-rule) that

$$s_r \cdot s_s = \sum_{c \geq \max(r-s,0)} s_{(s+c, r-c)},$$

where we assume that $s_{\lambda} = 0$ unless $\lambda$ is a partition with nonnegative parts.
Problem 13. Let $V$ be the (infinite dimensional) linear space with the basis $e_0, e_1, e_2, \ldots$. For $c \in \mathbb{Z}_{\geq 0}$, let $R(c)$ be the operator on $V \otimes V$ given by

$$R(c) : e_r \otimes e_s \mapsto \begin{cases} e_{s+c} \otimes e_{r-c} & \text{if } c \geq r - s \\ 0 & \text{otherwise} \end{cases}$$

(We assume that $e_i = 0$ for $i < 0$.) Let $R_{ij}(c)$ denote the operator that acts as $R(c)$ on the $i$-th and $j$-th copies of $V$ in the tensor power $V \otimes V \otimes V$. Show that the operator $R(c)$ satisfies the generalized Yang-Baxter equation:

$$R_{23}(c_{23})R_{13}(c_{13})R_{12}(c_{12}) = R_{12}(c'_{12})R_{13}(c'_{13})R_{23}(c'_{23}),$$

where

$$\begin{cases} c'_{12} = \min(c_{12}, c_{13} - c_{23}) \\ c'_{13} = c_{12} + c_{23} \\ c'_{23} = \max(c_{23}, c_{13} - c_{12}) \end{cases}$$

Problem 14. Let $A$ be a generic upper-triangular $n \times n$ matrix. Find the number of non-zero minors of $A$ of all sizes (including the empty minor of size $0 \times 0$).

Problem 15. 1. Find the bijective map $(x, y) \mapsto (x', y')$ from $\mathbb{R}_{>0}^2$ to $\mathbb{R}_{>0}^2$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x' & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

for some $t_1, t_2 \in \mathbb{R}_{>0}$.

2. Show that the double Bruhat cell $B_{u,w}$, $u, w \in S_n$, defined in terms of a double wiring diagram for $u$ and $w$ depends only on the permutations $u$ and $w$ (and not on a choice of a double wiring diagram).

Problem 16. Calculate the number of $d$-dimensional cells in the totally nonnegative Grassmannian $Gr_{\geq 0}(2, n)$.

Problem 17. Prove the equivalence of the 3 definitions of the strong Bruhat order on $S_n$:

A. Covering relations: $u < w$ iff $w = u \cdot (i, j)$ and $\ell(w) = \ell(u) + 1$.

B. $u \leq w$ if any reduced decomposition of $w$ has a subword which is a reduced decomposition of $u$.

C. $u \leq w$ if some reduced decomposition of $w$ has a subword which is a reduced decomposition of $u$.

Problem 18. Let $P$ be a path in any directed graph. Let us start erasing loops (i.e., closed directed paths without self-intersections) in $P$ until we get a path $P'$ without self-intersections. Is it true that the parity of the number of erased loops is a well-defined invariant of path $P$ and it does not depend on the order of erasing loops?