18.318 — Spring 2010 — Problem Set 2

Problem 1. (a) Let $K_{m,n}$ be the complete bipartite graph with vertices $1,\ldots,m$ (the left part) and $m+1,\ldots,m+n$ (the right part). For a spanning tree T of $K_{m,n}$, let Δ_T be the convex hull of the points $e_i+e_j \in \mathbb{R}^{m+n}$ for all edges (i,j) of T. A tree T is called *noncrossing* if it does not have a pair of edges (i,j) and (k,l) with $i<k<j<l$.

Show that the collection of simplices Δ_T, where T ranges over all noncrossing spanning trees of $K_{m,n}$, forms a triangulation of the product of two simplices $\Delta^{m-1}\times\Delta^{n-1}$.

(b) Construct a triangulation of $\Delta^2\times\Delta^2$ which cannot be obtained from the triangulation constructed in part (a) by a permutation the coordinates in $\mathbb{R}^{m+n}$.

(c) Describe all equivalence classes of triangulations of $\Delta^2\times\Delta^2$ under permutations of the coordinates.

Problem 2. Let $\rho=(\rho_1,\ldots,\rho_n)$ be a weakly increasing sequence of positive integers. A *ρ-parking function* is a sequence $(a_1,\ldots,a_n)$ of positive integers such that their increasing rearrangement $c_1\le\cdots\le c_n$ satisfies $c_i\le\rho_i$ for $i=1,\ldots,n$.

(a) Calculate the number of ρ-parking functions in the case when ρ_i is a linear function of i, that is $\rho=(l,l+k,l+2k,\ldots,l+(n-1)k)$.

(b) Let $I_\rho(q)=\sum_a q^{\sum(\rho_i-a_i)}$, where the sum is over all ρ-parking functions. Find a combinatorial interpretation of the value $I_\rho(-1)$ and prove it.

Problem 3. For positive integers n and k, the *generalized Shi arrangement* is the arrangement hyperplanes $\{x_i-x_j=r\}$ for $1\le i<j\le n$, $r=-k+1,-k+2,\ldots,k$.

Prove that the number of regions of the generalized Shi arrangement equals the number of ρ-parking functions for $\rho=(1,1+k,1+2k,\ldots,1+(n-1)k)$.

Problem 4. Fix positive integers n, k. Let S be the set of complex numbers $S = \{0\} \cup \{j \cdot \xi^r \mid j = 1, \dots, n;\ r = 0, \dots, k-1\}$, where $\xi = e^{2\pi\sqrt{-1}/n}$ is the primitive k-th root of 1. The cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts on S by multiplication by ξ. A tree T on $kn+1$ vertices labelled by the set S is called *k-symmetric* if it is invariant under this action of the cyclic group.

Prove that the number of k-symmetric trees on $nk+1$ vertices equals the number of ρ-parking functions for $\rho = (1, 1+k, 1+2k, \dots, 1+(n-1)k)$.

Problem 5. For $X = \{a_1, \dots, a_m\}$ where a_i's span $\mathbb{R}^d$, let I_X be the ideal in $\mathbb{C}[x_1, \dots, x_d]$ generated by the products of linear forms $\prod_{a\in Y} a(x)$ for all long subsets $Y \subset X$, and let P_X be the subspace of $\mathbb{C}[x_1, \dots, x_d]$ spanned by the products $\prod_{a\in Z} a(x)$ for all short subsets $Z \subset X$. (A subset $Y \subset X$ is called *long* (resp., *short*) if $X \setminus Y$ does not span (resp., spans) $\mathbb{R}^d$.

Prove that $\mathbb{C}[x_1, \dots, x_n] = I_X \oplus P_X$. (In class we proved 1/2 of this claim.)

Problem 6. Let S_x, $x \in P$, be a finite collection of subsets in some set A. Everybody knows the inclusion-exclusion formula:

$$|A \setminus \bigcup S_x| = |A| - \sum |S_x| + \sum |S_x \cap S_y| - \cdots$$

Suppose that the labelling set P is a poset, and the following condition holds: for any $x, y \in P$ there exists $z \in P$ such that $z \geq_P x$, $z \geq_P y$ and $S_x \cap S_y \subseteq S_z$. Show that in this case one can reduce the right-hand side of the inclusion-exclusion formula to a sum over strictly increasing chains in P:

$$|A \setminus \bigcup S_x| = |A| - \sum_x |S_x| + \sum_{x<y} |S_x \cap S_y| - \sum_{x<y<z} |S_x \cap S_y \cap S_z| + \cdots$$

Problem 7. Fix positive integers k, l, n. Let Π be the Pitman-Stanley polytope $\Pi = \{x \in \mathbb{R}^n| \mid x_i \geq 0;\ x_1 + \cdots + x_i \leq \rho_i,\ i = 1, \dots, n\}$ with $\rho_i = l + (i-1)k$.

Show that the Ehrhart polynomial of Π equals

$$i(\Pi, t) := \#\{t\Pi \cap \mathbb{Z}^n\} = \frac{1}{n!}(t\,l + 1)\prod_{i=2}^{n}(t(l+nk)+i).$$

Problem 8. Let $H_n(r)$ be the number of "magic squares", which are $n \times n$-matrices with nonnegative integer entries such that all row sums and all column sums are equal to r. Prove that $H_n(r)$ is a polynomial in r that has the following properties:

$$H_n(-1) = H_n(-2) = \cdots = H_n(-n+1) = 0$$

and $H_n(-n-r) = (-1)^{n-1} H_n(r)$.

...MORE PROBLEMS ...

Problem 9. Let $X = \{a_1, \ldots, a_m\}$ where a_i's span $\mathbb{R}^d$. Let I be the ideal in the ring of Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ generated by $\prod_{a \in Y}(1 - x^a)$ for all cocircuits $Y \subset X$. Let B be the subspace in $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ spanned by monomials x^b for all $b \in b(u)$, where $b(u) := (-Z_X + u) \cap \mathbb{Z}^d$ for some fixed generic vector $u \in \mathbb{R}^d$, and Z_X is the zonotope $Z_X := \sum_{a \in X}[0, a]$.

Show that $\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] = I \oplus B$.

Problem 10. Prove quasi-polynomiality and reciprocity of the Ehrhart polynomial $i(P, t)$ for a rational polytope P. (You may use the results about vector partition functions proved in class.)