18.315 PROBLEM SET 2 (due Friday, December 5, 2014)

1. In class we showed how to transform a SSYT into a pipe dream (aka RC-graph). Prove that this is a bijection between SSYTs corresponding to terms of the Schur polynomial $s_\lambda(x_1, \dots, x_k)$ and pipe dreams corresponding to terms of the Schubert polynomial $\mathfrak{S}_w$ for the Grassmannian permutation $w = w_\lambda$. Deduce that $\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \dots, x_k)$.

2. Construct an insertion procedure for pipe dreams such that

(a) It gives a combinatorial proof of Monk's Rule

$$(x_1 + \cdots + x_k)\, \mathfrak{S}_w = \sum_{i \le k < j,\, \ell(w\, t_{ij}) = \ell(w)+1} \mathfrak{S}_{w\, t_{ij}},$$

for a permutation $w \in S_n$ such that $w(n) = n$.

(b) For Grassmannian permutations $w = w_\lambda$ this insertion procedure is equivalent to the RSK insertion for SSYTs.

3. Give a bijective proof of the Cauchy formula for Schubert polynomials.

It is possible to obtain the Cauchy and/or the dual Cauchy formula for Schur polynomials as a special case of the Cauchy formula for Schubert polynomials?

4. A permutation $w \in S_n$ is called *fully commutative* if any two reduced decompositions for w can be obtained from each other by the commutation relations $s_i\, s_j = s_j\, s_i$ for $|i - j| \ge 2$.

(a) Show that w is fully commutative if an only if it is 321-avoiding, that is, there is no $i < j < k$ such that $w(i) > w(j) > w(k)$.

(b) Find the number of fully commutative permutations in S_n.

5. Let T_{ij} be the operator on $\mathbb{C}[x_1, x_2, \dots]$ such that, for $w \in S_\infty$,

$$T_{ij} : \mathfrak{S}_w \mapsto \begin{cases} \mathfrak{S}_{w\, t_{ij}} & \text{if } \ell(w\, t_{ij}) = \ell(w) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove the *Pieri Rule* for Schubert polynomials:

$$e_r(x_1, \dots, x_k)\, \mathfrak{S}_w = \sum T_{i_1 j_1} \cdots T_{i_r j_r}(\mathfrak{S}_w),$$

where the sum is over positive integers $i_1, \dots, i_r, j_1, \dots, j_r$ such that

1) $i_1, \dots, i_r \le k < j_1, \dots, j_r$,
2) $i_1, \dots, i_r$ are all distinct,
3) $j_1 \le j_2 \le \cdots \le j_r$.

6. Let $e_i^{(k)} := e_i(x_1, \dots, x_k)$ be the elementary symmetric polynomial in k variables.

(a) Prove the *Straightening Rule*

$$e_i^{(k)}\, e_j^{(k)} = e_i^{(k+1)}\, e_j^{(k)} + \sum_{l\geq 1} (e_{i-l}^{(k+1)}\, e_{j+l}^{(k)} - e_{i-l}^{(k)}\, e_{j+l}^{(k+1)}),$$

where we assume that $e_r^{(k)} = 0$ for $r < 0$.

(b) Deduce that the collection of polynomials $e_{i_1}^{(1)}\, e_{i_2}^{(m)} \cdots e_{i_m}^{(m)}$, where $m \geq 0$; $i_1, \dots, i_{m-1} \geq 0$, $i_m > 0$, is a linear basis of the polynomial ring $\mathbb{C}[x_1, x_2, \dots]$ in infinitely many variables.

7. Let $I_n = \langle e_1, \dots, e_n \rangle$ be the ideal in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ generated by the elementary symmetric polynomials $e_i = e_i(x_1, \dots, x_n)$. (Equivalently, I_n is generated by all symmetric polynomials in n variables without constant term.)

Fix the *degree-lex* ordering "$<$" of monomials in $\mathbb{C}[x_1, \dots, x_n]$, that is, $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if

$\sum a_i < \sum b_i$, or
$\sum a_i = \sum b_i$; and $a_j = b_j$, for $j < r$, and $a_r < b_r$, for some $r \in [n]$.

Prove that the reduced Gröbner basis of the ideal I_n with respect to the degree-lex term order is given by the complete homogeneous polynomials

$$h_k(x_1, \dots, x_{n+1-k}), \quad k = 1, \dots, n.$$

As a first step, you can prove the formula

$$h_k(x_1, \dots, x_l) = \det\left(e_{j-i+1}(x_1, \dots, x_{k+l-i})\right)_{i,j=1}^{k},$$

and use it to show that $h_k(x_1, \dots, x_{n+1-k})$ belong to the ideal I_n.

8. The *strong Bruhat order* on S_n can be defined by its covering relations $u \lessdot w$ if $w = u\, t_{ij}$ and $\ell(w) = \ell(u) + 1$. Prove that $u \leq w$ in the strong Bruhat order if and only if

$$r_{pq}(u) \geq r_{pq}(w), \text{ for all } p, q \in [n],$$

where $r_{pq}(w) = \#\{i \leq p \mid w(i) \leq q\}$.

9. Prove that

$$\sum_{w_0 = s_{i_1} \cdots s_{i_l}} i_1 \cdot i_2 \cdots i_l = \binom{n}{2}!$$

where the sum is over all reduced decompositions $w_0 = s_{i_1} \cdots s_{i_l}$ of the longest element w_0 in the symmetric group S_n.

10. Prove bijectively that $A_n = B_n = \binom{n}{2}!$, where

$A_n =$ the weighted sum over longest chains in the Hasse diagram of the *strong* Bruhat order on S_n, where the weight of an edge $w \to w\, t_{ij}$, $i < j$, is $j - i$.

$B_n =$ the weighted sum over longest chains in the Hasse diagram of the *weak* Bruhat order on S_n, where the weight of an edge $w \to w\, s_i$ is i.

11. Let $B = (b_{ij,k})$ be the $\binom{n}{2} \times n$-matrix with rows labelled by pairs $1 \leq i < j \leq n$ and columns labelled by $k = 1, \dots, n$ such that $b_{ij,k} = \delta_{i,k} - \delta_{j,k}$. Prove that the permanent of the matrix $B \cdot B^T$ equals

$$\mathrm{per}\,(B \cdot B^T) = 1!\,2!\,\cdots\,n!.$$

For example, for $n = 4$, we have

$$\mathrm{per}\left(\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}\right) = 1!\,2!\,3!\,4!.$$

Remark. The matrix $B^T \cdot B$ is the Laplacian matrix for the complete graph K_n. According to the *Matrix-Tree Theorem,* the determinant of the matrix obtained from $B^T \cdot B$ by removing any row and any column equals n^{n-2}. (This observation, is not really related to the problem above.)

12. (a) Prove that the following three conditions are equivalent:

(1) w is a *132-avoiding permutation,* that is, there is no $i < j < k$ such that $w(i) < w(k) < w(j)$.

(2) w is a *dominant permutation,* that is, its *Lehmer code* $(c_1, \dots, c_n)$, $c_i = \#\{j > i \mid w(j) < w(i)\}$, is weakly decreasing $c_1 \geq c_2 \geq \cdots \geq c_n$.

(3) The Schubert polynomial $\mathfrak{S}_w$ is equal to a single monomial.

(b) For a dominant permutation w, describe the Schubert polynomial $\mathfrak{S}_w$ in terms of w.

(c) Find the number of dominant permutations in S_n.

13. A permutation $w \in S_n$ is *strictly dominant* if its code $(c_1, c_2, \dots, c_n)$ is a strict partition, that is $c_1 > c_2 > \cdots > c_k = \cdots = c_n = 0$.

(a) Show that the following conditions are equivalent:

(1) w is strictly dominant.
(2) $w\, w_0$ is strictly dominant.
(3) both w and $w\, w_0$ are dominant.
(4) w is of the form $w_1 > w_2 > \cdots > w_k < w_{k+1} < \cdots < w_n$.
(5) w is both 132-avoiding and 231-avoiding.

(b) Find the number of strictly dominant permutations in S_n.

14. The *Schubert-Kostka* matrix is the $n! \times n!$ matrix $K = (K_{w,a})$ defined by $\mathfrak{S}_w(x) = \sum_a K_{w,a}\, x^a$, for $w \in S_n$. In other words, $K_{w,a}$ counts the number of pipe dreams for w of weight equal to x^a. Let $K^{-1} = (K^{-1}_{a,w})$ be the inverse matrix, that is $\sum_a K_{u,a} K^{-1}_{a,w} = \delta_{u,w}$ and $\sum_w K^{-1}_{a,w} K_{w,b} = \delta_{a,b}$.

(a) Let w be a strictly dominant permutation in S_n with code $(c_1 > c_2 > \cdots > c_k = \cdots = 0)$. Assume that $a = (a_1, \ldots, a_k, 0, \ldots, 0)$. Prove that

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(\sigma)} & \text{if } (a_1, \ldots, a_k) = (c_{\sigma_1}, \ldots, c_{\sigma_k}) \text{ for some } \sigma \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Assume that w is any 312-avoiding permutation. Prove that in this case $K^{-1}_{a,w}$ also equals 0, 1, or -1. Find the exact value of $K^{-1}_{a,w}$ in this case.

(c) Is it always true that $K^{-1}_{a,w} \in \{1, -1, 0\}$? Prove this or give a counterexample.

15. (a) Prove that the Schubert polynomial $\mathfrak{S}_w$ is equal to a single product of elementary polynomials $e^{(1)}_{i_1}\, e^{(2)}_{i_2} \cdots e^{(n-1)}_{i_{n-1}}$, for some $i_1, \ldots, i_{n-1}$, if and only if the permuation w avoids the patterns 312 and 1432.

(b) Find the number of permutations w in S_n that avoid both patterns 312 and 1432.

16. We constructed several linear bases of the *coinvariant algebra* $C_n = \mathbb{C}[x_1, \ldots, x_n]/I_n$:

$\{\mathfrak{S}_w \mid w \in S_n\}$,
$\{x_1^{a_1} \ldots x_n^{a_n} \mid 0 \le a_i \le n - i\}$,
$\{e^{(1)}_{i_1} \cdots e^{(n-1)}_{i_{n-1}} \mid 0 \le i_j \le j\}$,
$\{h^{(1)}_{i_1} \cdots h^{(n-1)}_{i_{n-1}} \mid 0 \le i_j \le n - j\}$.

Describe the bases of the space $H_n \simeq (C_n)^*$ of S_n-harmonic polynomial which are D-dual to these bases of C_n. (We defined the dual Schubert polynomials $\mathfrak{D}_w$ as elements of the basis dual to $\{\mathfrak{S}_w\}$. Describe the three other dual bases.)

17. (a) Prove that the space H_n of S_n-harmonic polynomials is spanned by partial derivatives $\prod(\partial/\partial y_i)^{a_i}$ of the Vandermonde determinant $\prod_{1\le i<j\le n}(y_i - y_j)$.

(b) Give a rule for the expansion of $(\partial/\partial y_i)\,\mathfrak{D}_w$ in terms of dual Schubert polynomials $\mathfrak{D}_u$.

(c) For a polynomial $f \in \mathbb{C}[y_1,\dots,y_n]$, let $d(f)$ be the dimension of the space spanned by partial derivatives $\left(\prod_{i=1}^n(\partial/\partial y_i)^{a_i}\right)(f)$, for all $a_1,\dots,a_n \ge 0$. Prove that $d(\mathfrak{D}_w)$ equals the size of the interval $[id,w]$ in the strong Bruhat order.

18. For two permutations $u \le w$ in the strong Bruhat order on S_n, define the polynomial

$$\mathfrak{D}_{u,w}(y_1,\dots,y_n) := \frac{1}{(\ell(w)-\ell(u))!}\sum_{u\lessdot\cdots\lessdot w}\prod_i weight(u^{(i-1)} \lessdot u^{(i)}),$$

where the sum is over saturated chains in the strong Bruhat order $u^{(0)} \lessdot u^{(1)} \lessdot \cdots \lessdot u^{(l)}$ from $u = u^{(0)}$ to $w = u^{(l)}$, and $weight(u \lessdot u\,t_{ij}) = y_i - y_j$, for $i<j$. In particular, $\mathfrak{D}_{id,w} = \mathfrak{D}_w$.

(a) Prove that

$$\mathfrak{D}_{u,w} = \sum_{v\in S_n} c_{u,v}^w\,\mathfrak{D}_v,$$

where $c_{u,v}^w$ are the *generalized Littlewood-Richarson coefficients* given by

$$\mathfrak{S}_u\,\mathfrak{S}_v = \sum_{w\in S_n} c_{u,v}^w\,\mathfrak{S}_w.$$

(b) Find a combinatorial expression for $d(\mathfrak{D}_{u,v})$, where $d(f)$ is defined in Problem 17(c).

19. The quantum cohomology ring of the Grassmannian $QH^*(Gr(k,n))$ is isomorphic to the quotient ring $\mathbb{C}[q][x_1,\dots,x_k]^{S_k}/J_{kn}^q$, where J_{kn}^q is the ideal

$$J_{kn}^q := \left\langle h_{n-k+1}, h_{n-k+2},\dots,h_{n-1}, h_n + (-1)^k q\right\rangle.$$

Prove the *Rim-Hook Algorithm* by showing that

(a) For a partition $\nu = (\nu_1,\dots,\nu_k)$, and a partition $\tilde\nu$ obtained from ν by removing a rim hook of size n and height h, we have

$$s_\nu(x_1,\dots,x_k) \equiv (-1)^{k-h} q\, s_{\tilde\nu}(x_1,\dots,x_k) \pmod{J_{kn}^q}.$$

(b) If $\nu \not\subseteq k\times(n-k)$ and there is no valid way to remove a rim hook of size n from ν, then $s_\nu(x_1,\dots,x_k) \equiv 0 \pmod{J_{kn}^q}$.