18.315 Problem Set 3 (due Tuesday, December 7, 2010)

**1.** The *coinvariant algebra* is $C_n := \mathbb{C}[x_1, \dots, x_n]/I_n$, where $I_n = \langle e_1, \dots, e_n \rangle$ is the ideal generated by the elementary symmetric polynomials $e_i = e_i(x_1, \dots, x_n)$, $i = 1, \dots, n$. Show that $C_n$ is an $n!$-dimensional vector space with a linear basis given by the cosets of the monomials $x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ with $a_i \le n - i$ for all $i$.

**2.** Find the number of permutations $w \in S_n$ such that any two reduced decompositions for $w$ can be related to each other by a sequence of 2-moves.

**3.** Let $w$ be any permutation in $S_n$ and $w_0$ be the longest permutation in $S_n$. Show that $\mathfrak{S}_w(x_1, \dots, x_n) \equiv \mathfrak{S}_{w_0\, w\, w_0}(-x_n, \dots, -x_1)$ modulo the ideal $I_n$ from problem 1.

**4.** Let $h_i(x) := 1 + x\, u_i$, $i = 1, \dots, n-1$, where the $u_i$ are the generators of the nilHecke algebra and $x$ commutes with all $u_i$'s. The $h_i(x)$ satisfy the Yang-Baxter relations. Let

$$\phi_n(x, y) := \prod_{i=1}^{n-1} \prod_{j=n-i}^{i} h_{i+j-1}(x_i - y_j).$$

Use the Yang-Baxter relations to show that $\phi_n(x, y) = \phi_n(0, y) \cdot \phi_n(x, 0)$.

**5.** (A) In class we constructed the insertion procedure for RC-graphs. Show that it is well-defined and invertible. Deduce Monk's rule.

(B) Show that, in case of RC-graphs for Grassmannian permutations, this procedure is equivalent to the RSK insertion for SSYTs.

**6.** Let us say that a permutation $w \in S_n$ is *strictly dominant* if its code $code(w) = (c_1, c_2, \dots, c_n)$ is a strict partition, that is $c_1 > c_2 > \cdots > c_k = \cdots = c_n = 0$.

(A) Show that the following conditions are equivalent:
(1) $w$ is strictly dominant.
(2) $w\, w_0$ is strictly dominant.
(3) $w$ is of the form $w_1 > w_2 > \cdots > w_k < w_{k+1} < \cdots < w_n$.
(4) $w$ is both 132-avoiding and 231-avoiding.

(B) Find the number of strictly dominant permutations in $S_n$.

**7.** The *Schubert-Kostka* matrix is the $n! \times n!$ matrix $K = (K_{w,a})$ defined by $\mathfrak{S}_w(x) = \sum_a K_{w,a}\, x^a$, for $w \in S_n$. In other words, $K_{w,a}$

counts the number of RC-graphs for $w$ of $x$-weight equal to $x^a$. Let $K^{-1} = (K^{-1}_{a,w})$ be the inverse matrix, that is $\sum_a K_{u,a} K^{-1}_{a,w} = \delta_{u,w}$ and $\sum_w K^{-1}_{a,w} K_{w,b} = \delta_{a,b}$.

(A) Let $w$ be a strictly dominant permutation in $S_n$ with code $(c_1 > c_2 > \cdots > c_k = \cdots = 0)$. Assume that $a = (a_1, \ldots, a_k, 0, \ldots, 0)$. Prove that

$$K^{-1}_{a,w} = \begin{cases} (-1)^{\ell(\sigma)} & \text{if } (a_1, \ldots, a_k) = (c_{\sigma_1}, \ldots, c_{\sigma_k}) \text{ for some } \sigma \in S_k, \\ 0 & \text{otherwise.} \end{cases}$$

(B) Now assume that $w$ is any 312-avoiding permutation. Prove that in this case $K^{-1}_{a,w}$ also equals 0, 1, or $-1$. Find the exact value of $K^{-1}_{a,w}$ in this case.

(C) Is it always true that $K^{-1}_{a,w} \in \{1, -1, 0\}$?