18.315 PROBLEM SET 2 (due Tuesday, November 30, 2010)

1. Let P be a poset. Let C_k (resp., A_k) be the maximal number of elements of P that can be covered by k chains (resp., antichains). Prove that $\lambda = (C_1, C_2 - C_1, C_3 - C_2, \dots)$ and $\mu = (A_1, A_2 - A_1, A_3 - A_2, \dots)$ are two partitions conjugate to each other.

2. Fix three partitions λ, μ, ν such that $|\nu| = |\lambda| + |\mu|$ and two SSYTs U_0 of shape μ and V_0 of shape ν. Consider the following two sets. The first set is the set of pairs (T, S) of SSYTs such that $sh(T) = \lambda$, $sh(S) = \mu$ and $T \cdot S = V_0$. The second set is the set of SSYTs U of skew shape ν/λ such that $jdt(U) = U_0$. In class we constructed a map from the first set to the second set using RSK. Prove that this map is a bijection.

3. Let $TSSCPP_n$ be the number of a *totally symmetric self-complementary plane partitions* that fit inside the $2n \times 2n \times 2n$ box, that is, order ideals in the poset $[2n] \times [2n] \times [2n]$ (the product of three chains) which are symmetric with respect to the S_3-action and with respect to taking the complement. Let ASM_n be the number of $n \times n$ *alternating sign matrices,* which are matrices filled with 0's, 1's, and -1's such that in each row and in each column the nonzero entries are arranged as $1, -1, 1, -1, \dots, -1, 1$ (an alternating sequence that starts and ends with 1). Prove that $TSSCPP_n = ASM_n$.

4. Fix positive integers k, l, m. Prove that

$$\prod_{a=1}^{k}\prod_{b=1}^{l}\prod_{c=1}^{m} \frac{[a+b+c-1]_q}{[a+b+c-2]_q} = \prod_{x \in [k]\times[l]} \frac{[c(x)+m+k]_q}{[h(x)]_q},$$

where $c(i,j) = j - i$, $h(i,j) = k - i + l - j + 1$, and $[n]_q := (1-q^n)/(1-q)$.

5. Consider reduced decompositions of the longest permutation w_0 in S_n. Let us identify the reduced decompositions that can be obtained from each other by 2-moves, the equivalence classes are called *commutation classes.* Prove that in each commutation class there are at least $n-2$ possibilities to make a 3-move.

6. Show that Schützenberger's jeu de taquin is equivalent to growth diagrams with the local rules given in class.

7. Let $LR(\lambda/\mu, \nu)$ be the set of Littlewood-Richardson tableaux of shape λ/μ and weight ν. Find bijections between the following sets

$LR(\lambda/\mu, \nu)$, $LR(\lambda/\nu, \mu)$, $LR(\lambda'/\mu', \nu')$, $LR(\mu^\vee/\lambda^\vee, \nu)$, $LR(\nu^\vee/\lambda^\vee, \mu)$. For the last two sets we assume that all λ, μ, ν fit inside the $k \times l$ rectangle; and $\lambda^\vee$ denotes the complement $(k \times l)/\lambda$ rotated 180°.

8. Prove the rule for the inner product $\langle s_{\lambda/\mu},\ s_{\nu/\gamma}\rangle$ of skew Schur functions in terms of Zelevinsky's pictures. Show that, in the case when $\gamma = \emptyset$, the Zelevinsky picture rule is equivalent to the classical Littlewood-Richardson rule.

9. In class we showed how to transform a Littlewood-Richardson tableau T into a BZ-pattern P. (The entries of P are expressed in terms the Gelfand-Tsetlin coordinates of T. Honeycombs help to describe this transformation.) Verify that, under this transformation, the definition of Littlewood-Richardson tableaux is equivalent to the definition of BZ-patterns.

10. Construct a bijection between BZ-patterns and BZ-triangles. Recall that BZ-patterns satisfy the hexagon condition, and BZ-triangles satisfy the tail-nonnegativity condition.

11. (A) Prove Gleizer-Postnikov's rule for $c_{\lambda\mu}^{\nu}$ in terms of integer web diagrams. Recall that these diagrams are similar to Knutson-Tao's honeycombs, but have infinite rays of 4 different directions, instead of 3 directions in Knutson-Tao's honeycombs. Construct a bijection between KT honeycombs and GP web diagrams.

(B) Investigate the number of integer web diagrams with fixed infinite rays in 5 different directions.

12. (A) Let $GT(\lambda, \mu)$ be the Gelfand-Tsetlin polytope, that is the polytope of real-valued Gelfand-Tsetlin patterns. Find two integer partitions λ, μ such that $GT(\lambda, \mu)$ has a non-integer vertex.

(B) Let $BZ(\lambda, \mu, \nu)$ be the Berenstein-Zelevinsky polytope, that is the polytope of real valued BZ-patterns. Fix two partitions λ and μ with n parts. Show that, for a sufficiently large number M, we have $GT(\lambda, \mu) = BZ(\nu, \mu, \lambda + \nu)$ for any partition ν with n parts such that $\nu_i - \nu_{i+1} > M$ for $i = 1, \ldots, n-1$.

(C) Find three integer partitions λ, μ, ν such that the Berenstein-Zelevinsky polytope $BZ(\lambda, \mu, \nu)$ has a non-integer vertex. Draw the honeycomb corresponding to this non-integer vertex. Does this honeycomb have a cycle?

13. Let $\chi_{\lambda\mu}$ be the coefficients in the Schur expansion of power symmetric functions: $p_\mu = \sum_\lambda \chi_{\lambda\mu}\, s_\lambda$. Prove that $s_\lambda = \sum_\mu \chi_{\lambda\mu} (z_\mu)^{-1}\, p_\mu$, where $z_\mu = \prod_{i\geq 1} i^{m_i}\, m_i!$ and m_i is the number of parts of μ equal to i.

14. Prove that the Schur function s_λ in two infinite sets of variables equals

$$s_\lambda(x_1, x_2, \ldots, y_1, y_2, \ldots) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu(x_1, x_2, \ldots)\, s_\nu(y_1, y_2, \ldots),$$

where $c^\lambda_{\mu\nu}$ is the Littlewood-Richardson coefficient.

15. Find a probabilistic hook-walk proof the hook-length formula for the number of SYTs of a shifted shape.