18.315 Problem Set 1 (due Tuesday, October 19, 2010)

1. Let B_{n+1} be the number of sequences $\lambda^{(0)}, \dots, \lambda^{(2n)}$ of Young diagrams such that:

(1) $\lambda^{(0)} = \lambda^{(2n)} = \emptyset$.
(2) For $i = 1, \dots, n$, $\lambda^{(2i-1)}$ either equals $\lambda^{(2i-2)}$ or is obtained from $\lambda^{(2i-2)}$ by adding a corner box.
(3) For $i = 1, \dots, n$, $\lambda^{(2i)}$ either equals $\lambda^{(2i-1)}$ or is obtained from $\lambda^{(2i-1)}$ by removing a corner box.

Show that B_{n+1} is the *Bell number,* that is, the number of set-partitions of $[n{+}1]$. Can you construct a bijection between sequences $\lambda^{(0)}, \dots, \lambda^{(2n)}$ and set-partitions?

2. Let P_n be the number of lattice paths P in $\mathbb{Z}^2$ from $(0,0)$ to $(2n,0)$ such that:

(1) P consists of the steps $(1,1)$, $(1,-1)$, and $(-1,1)$.
(2) P always stays in the first quadrant $\mathbb{Z}^2_{\geq 0}$;
(3) P never visits the same point twice.

Prove that $P_n = (2n-1)!!$.

3. In class we constructed two bijective piecewise linear continuous maps ϕ_{RSK} and ϕ_{HG} from the set of all nonnegative integer $n \times n$ matrices to the set reverse plane partitions of shape $n \times n$ (that is, nonnegative integer matrices weakly increasing in rows and columns). The first map ϕ_{RSK} is the RSK correspondence (written in terms of Gelfand-Tsetlin patterns) and the second map ϕ_{HG} is the inverse Hillman-Grassl correspondence.

(A) Write down explicit formulas (in terms of min and max) for these maps for $n = 3$.

(B) Write down explicit formulas for these maps for $n = 4$.

(C) Investigate the relation between ϕ_{RSK} and ϕ_{HG}.

4. Prove the following identity

$$\sum_{\lambda} s_\lambda(x_1, x_2, \dots)\, s_{\lambda'}(y_1, y_2, \dots) = \prod_{i,j \geq 1} (1 + x_i y_j).$$

5. For two fixed partitions λ and μ, show that

$$\sum_{\nu \supseteq \lambda \cup \mu} s_{\nu/\lambda}(x)\, s_{\nu/\mu}(y) = \left(\sum_{\gamma \subseteq \lambda \cap \mu} s_{\lambda/\gamma}(x)\, s_{\mu/\gamma}(y) \right) \prod_{i,j} (1 - x_i y_j)^{-1}.$$

6. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z} \setminus \{0\})^n$ be an integer sequence with zero sum. Let $(\beta_1, \dots, \beta_p)$ be the positive entries of α, and $(\gamma_1, \dots, \gamma_q)$ be the minus negative entries of α. (Here $n = p + q$.) Let μ be the Young diagram with p columns and q rows such that the path P from the bottom left corner of μ to the top right corner of μ is given by the rule: if $\alpha_i > 0$ (resp., $\alpha_i < 0$) then the ith step in P is $(1, 0)$ (resp., $(0, 1)$).

Construct a bijection $\phi : A \to B$ between the following two sets. The set A is the set of sequences $\lambda^{(0)}, \dots, \lambda^{(n)}$ of Young diagrams such that:

(1) $\lambda^{(0)} = \lambda^{(n)} = \emptyset$.
(2) If $\alpha_i > 0$, then $\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal α_i-strip.
(3) If $\alpha_i < 0$, then $\lambda^{(i-1)}/\lambda^{(i)}$ is a *vertical* $(-\alpha_i)$-strip.

The set B is the set of fillings of the shape μ with 0's and 1's such that the column sums are $\beta_1, \dots, \beta_p$ (from left to right), and the row sums are $\gamma_1, \dots, \gamma_p$ (from bottom to top).

7. Prove that $(\sum'_\mu s_\mu)(e_0 + e_1 + e_2 + \dots) = \sum_\lambda s_\lambda$, where $\sum'_\mu$ is the sum over partitions μ with all even parts, and the sum in the right-hand side is over all partitions λ.

8. Show that Fomin's growth diagrams are related to RSK, as explained in class.

9. Prove that Viennot's shadow construction is related to RSK, as explained in class.

10. The elements of the Fibonacci lattice can be labelled by compositions c with all parts equal to 1 or 2.

(A) Describe the covering relation of the Fibonacci lattice nonrecursively in terms 1-2-compositions c.

(B) For a 1-2-composition c, let f^c be the number of increasing paths in the Fibonacci lattice from $\hat{0}$ to c. Investigate the numbers f^c. Is there a hook-length formula for f^c?

11. Construct an analogue of Fomin's growth diagrams for the Fibonacci lattice.

12. Let $\lambda = (\alpha_1, \dots, \alpha_k \mid \beta_1, \dots, \beta_k)$ (in Frobenius notation). Prove Giambelli's formula $s_\lambda = \det(s_{(\alpha_i|\beta_j)})_{i,j=1}^k$. Can you give a proof based on Lindström's lemma?