Exam 2

November 04, 2005

You have 1 hour 20 min to solve the following problems. The problems worth 10 points each. You can use your notes, books, calculators, etc. Show your reasoning.

1. Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 3a_n$, $n \geq 0$, with the initial conditions $a_0 = 0$ and $a_1 = 2$.

2. Solve the recurrence relation $a_n = n^2\, a_{n-1} + (n^2 - 1)$, $n \geq 1$, with the initial condition $a_0 = 0$.

3. Calculate the number of $n \times n$ matrices filled with 0's and 1's such that the first row, the last row, the first column, and the last column contain some nonzero entries.

4. Let a_n be the number of compositions of n into parts of size 1 or 2 with parts of size 2 colored in 2 colors. For example, $a_3 = 5$, corresponding to the colored compositions $(1,1,1)$, $(1,2)$, $(1,\mathbf{2})$, $(2,1)$, $(\mathbf{2},1)$.

   (a) Find a recurrence relation for the numbers a_n.

   (b) Find the generating function $\sum_{n\geq 0} a_n x^n$. (Assume that $a_0 = 1$.)

   (c) Find an explicit formula for a_n

5. Let $f(n)$ be the number of ways subdivide n people into groups with at least 2 members and then select a group leader in each group. For example, $f(4) = 16$, corresponding to the following set partitions with group leaders marked in bold: **1**234, **2**134, **3**124, **4**123, **1**2|**3**4, **1**2|**4**3, **2**1|**3**4, **2**1|**4**3, **1**3|**2**4, **1**3|**4**2, **3**1|**2**4, **3**1|**4**2, **1**4|**2**3, **1**4|**3**2, **4**1|**2**3, **4**1|**3**2. Assume that $f(0) = 1$.

   (a) Find the exponential generating function $\sum_{n\geq 0} f(n)\, x^n/n!$.

   (b) Calculate $f(5)$, $f(6)$, and $f(7)$. (Hint: See next page.)