Exam 2
November 04, 2005

You have 1 hour 20 min to solve the following problems. The problems worth 10 points each. You can use your notes, books, calculators, etc. Show your reasoning.

1. Solve the recurrence relation $a_{n+2} = 4a_{n+1} - 3a_n$, $n \geq 0$, with the initial conditions $a_0 = 0$ and $a_1 = 2$.

2. Solve the recurrence relation $a_n = n^2 a_{n-1} + (n^2 - 1)$, $n \geq 1$, with the initial condition $a_0 = 0$.

3. Calculate the number of $n \times n$ matrices filled with 0’s and 1’s such that the first row, the last row, the first column, and the last column contain some nonzero entries.

4. Let $a_n$ be the number of compositions of $n$ into parts of size 1 or 2 with parts of size 2 colored in 2 colors. For example, $a_3 = 5$, corresponding to the colored compositions $(1, 1, 1), (1, 2), (1, 2), (2, 1), (2, 1)$.
   (a) Find a recurrence relation for the numbers $a_n$.
   (b) Find the generating function $\sum_{n \geq 0} a_n x^n$. (Assume that $a_0 = 1$.)
   (c) Find an explicit formula for $a_n$

5. Let $f(n)$ be the number of ways subdivide $n$ people into groups with at least 2 members and then select a group leader in each group. For example, $f(4) = 16$, corresponding to the following set partitions with group leaders marked in bold: \textbf{1}234, \textbf{2}134, \textbf{3}124, \textbf{4}123, 12\textbf{3}4, 12\textbf{4}3, 21\textbf{3}4, 21\textbf{4}3, 13\textbf{2}4, 13\textbf{4}2, 31\textbf{2}4, 31\textbf{4}2, 14\textbf{2}3, 14\textbf{3}2, 41\textbf{2}3, 41\textbf{3}2. Assume that $f(0) = 1$.
   (a) Find the exponential generating function $\sum_{n \geq 0} f(n) \frac{x^n}{n!}$.
   (b) Calculate $f(5), f(6), f(7)$. (Hint: See next page.)