PROBLEM SET 3 (due on Thursday 10/21/2004)

The problems worth 10 points each.

Problem 1 Let $c(n, k)$ be the signless Stirling number of the first kind. Show that
$$\sum_{k=0}^{n} 2^k \, c(n, k) = (n + 1)!.$$
What can you say about $\sum_{k=0}^{n} 2^k \, s(n, k)$, where $s(n, k) = (-1)^{n-k} c(n, k)$?

Problem 2 Show that $\sum_{x=1}^{n} x(x-1)(x-2)\cdots(x-k+1) = k! \binom{n+1}{k+1}$. Use this identity to deduce a general formula for the sum $1^k + 2^k + \cdots + n^k$ for an arbitrary n and k in terms of the Stirling numbers of the second kind. Specialize this formula for $k = 5$ and write an explicit closed expression for $1^5 + \cdots + n^5$.

Problem 3 Find the number of integers $i \in \{1, \ldots, 1000\}$ that are not divisible by 3, 5, or 7.

Problem 4 As you know, the primes less than 10 are 2, 3, 5, and 7. Use this fact and the Inclusion-Exclusion to find the number of primes less than 100.

Problem 5 Let D_n be the *derangement number*, i.e., D_n is the number of permutations $w \in S_n$ such that $w_i \neq i$, for $i = 1, \ldots, n$. Also let Q_n be the number of permutations $u \in S_n$ such that $u_{i+1} \neq u_i + 1$, for $i = 1, \ldots, n-1$.

(A) Show that $Q_n = D_n + D_{n-1}$.
(B) Show that $D_{n+1} = n \cdot Q_n$.
(C) Deduce the recurrence relations
$$D_{n+1} = n(D_n + D_{n-1}),$$
$$Q_{n+1} = nQ_n + (n-1)Q_{n-1}.$$

Can you prove any of these identities combinatorially?

Problem 6 Let $f_n(k)$ be the number of permutations in S_n with exactly k fixed points. (Recall that a fixed point in $w \in S_n$ is an index i such that $w_i = i$.)

(A) Express $f_n(k)$ in terms of the derangement numbers.
(B) Find the median
$$M_n = \frac{1}{n!} \sum_k k \, f_n(k)$$
of the distribution $w \mapsto \#\{\text{fixed points in } w\}$. In other words, M_n is the average number of fixed points in a random permutation in S_n.

(C)* Let $p_n(k)$ be the probability that a randomly chosen permutation $w \in S_n$ (with the uniform distribution) has k fixed points. Find the limit $p(k) = \lim_{n\to\infty} p_n(k)$. You can think of $p(k)$ as the probability that a "random infinite permutation" has k fixed points. Can you recognize the distribution $p(k)$?

Problem 7 Which of the following two numbers is bigger: the number of permutations in S_n without fixed points or the number of permutations in S_n with exactly one fixed point?

Problem 8 Find the number of integer solutions of the following system of equations and inequalities:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10,$$
$$0 \leq x_i \leq 3, \quad \text{for } i = 1, \ldots, 5.$$

Problem 9 At a party that consists of 7 couples all people should to be seated at a round table so that no two spouses (or partners) are allowed to sit next to each other. In how many ways can these 14 people be seated?

Problem 10 Suppose that a sequence a_n satisfies the "anti-Fibonacci" recurrence relation: $a_n = a_{n-1} - a_{n-2}$. Show that the sequence a_n is periodic with period 6, i.e., $a_{n+6} = a_n$.

Bonus Problems

Problem 11 (*) This problem generalizes Problem 6(B). Let G be a subgroup of the symmetric group S_n. And let M_G the average number of fixed points of elements of G:

$$M_G = \frac{1}{|G|} \sum_{g \in G} \#\{i \in \{1, \ldots, n\} \mid g(i) = i\}.$$

(A) Assume that, for any $i, j \in \{1, \ldots, n\}$, there exists an element $g \in G$ such that $g(i) = j$. Find an explicit expression for M_G. Give a combinatorial proof.

(B) Find M_G for any subgroup in S_n.

Problem 12 (*) Let E_n be the sequence of rational numbers defined by $E_0 = E_1 = 1$ and $E_{n+1} = (1 + E_n^2)/E_{n-1}$, for $n \geq 1$. (From this definition it not even clear that the E_n are integers.) Show that $E_n = F_{2n}$, where F_k are the Fibonacci numbers.