

## PRACTICE HOUR EXAM #2

for the Hour Exam of November 12, 2003

**Disclaimer.** This practice exam will give you an idea of what the actual Hour Exam is like, but the actual Hour Exam need not have individual problems which are parallel to or analogous to problems on this practice exam.

Closed book, notes, calculators, computers, etc. Do all four problems. There are a total of 40 points.

1. (a) (5 points) Find polynomials  $F(x)$  and  $G(x)$  such that

$$\sum_{n \geq 0} \binom{n}{2} x^n = \frac{F(x)}{G(x)}.$$

**Hint.** One way to do this is to start with the identity  $\sum_{n \geq 0} x^n = 1/(1-x)$ .

- (b) (5 points) There are a line of  $n$  trees. Some workpersons put in some road signs between the trees. Between any two consecutive tree can be at most one sign. Every sign must be between two trees. In each maximal line of trees not separated by signs (including the beginning and end lines) the workpersons paint two of the trees red. Let  $f(n)$  be the number of ways this can be done. Find polynomials  $A(x)$  and  $B(x)$  such that

$$\sum_{n \geq 0} f(n)x^n = \frac{A(x)}{B(x)}.$$

**Example.** When  $n = 3$  no signs are possible since that would create a line of one tree, and thus two of them can't be painted red. Hence  $f(3) = \binom{3}{2} = 3$ . When  $n = 4$ , there are 6 ways if no signs are placed, and one way if a sign is place in the middle. Hence  $f(4) = 6 + 1 = 7$ .

2. (10 points) Solve the recurrence  $a_{n+1} = 2a_n + n$ ,  $n \geq 0$ , with the initial condition  $a_0 = 0$ .

3. (10 points) Let  $f(n)$  be the number of ways to choose a permutation  $\pi$  of  $1, 2, \dots, n$  and then color each fixed point of  $\pi$  either red or blue. Find a simple expression for the exponential generating function

$$F(x) = \sum_{n \geq 0} f(n) \frac{x^n}{n!}.$$

**Example.** One permutation of  $1, 2, 3$  has three fixed points, so there are  $2^3 = 8$  colorings of its fixed points. Three permutations have one fixed point, with two colorings each. Two permutations have no fixed points, with one coloring each. Thus  $f(n) = 8 + 3 \cdot 2 + 2 \cdot 1 = 16$ .

4. (10 points) A connected graph  $G$  on  $n$  vertices has exactly one cycle. How many edges does  $G$  have? (Give a proof.)