PROBLEM SET 3 (due on Thursday 05/16/2013)

Solve as many problems as you like. Hand in at least 3 problems.

Problem 1. Show that the numbers of alternating permutations $A(n) := \#\{w \in S_n \mid w_1 < w_2 > w_3 < w_4 > \cdots\}$ are given by the Euler-Bernoulli triangle, as described in class.

Problem 2. Consider a pair (P_1, P_2) of lattice paths on the plane such that

(1) Both P_1 and P_2 start at the origin, have n steps $(1, 0)$ (right) or $(0, 1)$ (up), and end at the same point.

(2) The path P_2 is located to the South-East of P_1, and it touches P_1 only at the end-points.

Find the number of pairs (P_1, P_2).

Problem 3. Prove that all cofactors of a square matrix with zero row and column sums are equal to each other. (The ij-th cofactor is $(-1)^{i+j}$ times the determinant of the matrix with removed i-th row and j-th column.)

Problem 4. Prove the following reciprocity formula for spanning trees.

Let G be a graph on the vertices $1, \dots, n$ (without loops and multiple edges); and let $\bar{G}$ be its *complementary graph* on the same vertices $1, \dots, n$. This means that two vertices i and j, $i \neq j$, are connected by an edge in $\bar{G}$ if and only if they are not connected by an edge in G, and vise versa.

Let G^+ be the graph obtained from G by adding a new vertex 0 and connecting it with all vertices of G. Define the polynomial

$$T_G(x_0, x_1, \dots, x_n) := \sum_T \prod_{i=0}^{n} x_i^{\deg_T(i)-1},$$

where the sum is over all spanning trees T of the extended graph G^+, and $\deg_T(i)$ denotes the degree of vertex i in T. Similarly, define the polynomial $T_{\bar{G}}(x_0, x_1, \dots, x_n)$ for the complementary graph $\bar{G}$.

Prove the formula

$$T_{\bar{G}}(x_0, x_1, x_2, \dots, x_n) = (-1)^{n-1} T_G(-x_0 - x_1 - \cdots - x_n, x_1, x_2, \dots, x_n).$$

Problem 5. Use the reciprocity formula from the previous problem to give

(a) a short proof of Cayley's formula for the number of spanning trees in the complete graph K_n,

(b) a formula for the number of spanning trees in the complete bipartite graph $K_{m,n}$,

(c) a formula for the number of spanning trees in the complete tripartite graph $K_{m,n,k}$. (The complete tripartite graph $K_{m,n,k}$ is the graph whose vertices are subdivided into 3 parts of sizes m, n, k, and two vertices are connected by an edge if and only if they are in different parts.)

Problem 6. Give bijective proofs of the formulas for the number of spanning trees in (a) $K_{m,n}$, and (b) $K_{m,n,k}$.

Problem 7. Consider the electrical network given by the edges of the n-hypercube with all edge resistances equal to 1. Find the effective resistance between a pair of the opposite vertices of the hypercube.

Problem 8. Let G be a graph with some vertices marked by $A_1, \dots, A_n$. (Some vertices of G can be unmarked.)

For a set-partition $\pi = \pi_1|\pi_2|\cdots|\pi_k$ of the set $[n]$ with k blocks, let $f(\pi)$ be the number of subforests F in the graph G with exactly k connected components such that each vertex of G belongs to one of the components of F, and two marked vertices A_i and A_j are in the same component of F if and only if i and j are in the same block of π.

The numbers $f(\pi)$ are important for electrical networks. For example, we proved in class, that for $n = 2$, the effective resistance between A_1 and A_2 is $\frac{f(1|2)}{f(12)}$.

Prove the following relation for these numbers, for $n = 3$,

$$f(123)f(1|2|3) = f(12|3)f(23|1) + f(13|2)f(12|3) + f(23|1)f(13|2).$$

For example, if $G = K_3$ with 3 marked vertices, then $f(123) = 3$, $f(1|2|3) = 1$, $f(12|3) = f(13|2) = f(23|1) = 1$.

Also, if G is the Y-shaped graph with 3 marked end-points (and unmarked central vertex), then $f(123) = 1$, $f(1|2|3) = 3$, $f(12|3) = f(13|2) = f(23|1) = 1$. In both examples, the relation holds.

Problem 9. Let $I_n(x) = \sum_T x^{\mathrm{inv}(T)}$, where the sum is over trees T on $n+1$ nodes labelled $0, 1, \dots, n$ and $\mathrm{inv}(T)$ denotes the number of *inversions* in T, that is, pairs (i, j), $1 \le i < j \le n$, such that j belongs to the shortest path from i to 0.

Also let $P_n(x) = \sum_f x^{\binom{n+1}{2} - \sum f(i)}$, where the sum is over parking functions $f : [n] \to [n]$ of size n.

Prove that $I_n(x) = P_n(x)$.

Problem 10. Fix positive integers n, r, l. We say that a function $f : [n] \to [nr + l - 1]$ is a *generalized parking function* of type (n, r, l) if it satisfies the condition $\#i \mid f(i) \ge (n-k)r + l \le k$ for $k = 0, 1, \dots, n$.

Find a closed formula for the number of such generalized parking functions.

Problem 11. Fix n, r. Consider the action of the cyclic group $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$ on the set $[nr]$ such that the generator of $\mathbb{Z}_r$ sends i to $i + n \pmod{nr}$; and extend this action to the set $[nr] \cup \{0\}$, where 0 is a fixed point of the action. We say that a tree T on $nr+1$ nodes labelled $0, 1, 2, \dots, nr$ is a *$\mathbb{Z}_r$-invariant tree* if it is invariant under the above action of the cyclic group on its nodes.

(a) Find a formula for the number of $\mathbb{Z}_r$-invariant trees on $nr + 1$ nodes.

(b) Construct a bijection between $\mathbb{Z}_r$-invariant trees on $nr + 1$ nodes and generalized parking functions of type (n, r, l) (as in the previous problem) with $l = 1$.

Problem 12. Calculate the number of regions of the following hyperplane arrangements in $\mathbb{R}^n$. (Here $x_1, \dots, x_n$ are the coordinates in $\mathbb{R}^n$.)

(a) $x_i - x_j = 0$, for $1 \le i < j \le n$ $\quad\binom{n}{2}$ hyperplanes

(b) $x_i - x_j = 0, 1$, for $1 \le i < j \le n$ $\quad 2\binom{n}{2}$ hyperplanes

(c) $x_i - x_j = -1, 0, 1$, for $1 \le i < j \le n$ $\quad 3\binom{n}{2}$ hyperplanes

Problem 13. Recall that the *Minkowski sum* of two subsets A and B in $\mathbb{R}^n$ is $A + B := \{a + b \mid a \in A, b \in B\}$.

Let us also define the *Minkowski difference* $A - B := \{c \in \mathbb{R}^n \mid c + B \subseteq A\}$. Here $c + B$ denotes the parallel translation of B by the vector c. (If there is no parallel translation of B that fits inside A, then the Minkowski difference $A - B$ is the empty set.)

(a) Is it true that, for any convex polytopes A and B, we have $(A+B)-B = A$? (Give a proof or a counterexample)

(b) Is it true that, for any convex polytopes A and B, we have $(A-B)+B = A$? (Give a proof or a counterexample)

(c) We defined *zonotopes* as Minkowski sums of line intervals. Will we get a bigger class of polytopes, if we consider Minkowski sums *and differences* of line intervals?

(d) Is it true that a zonotope has a *unique* expression as a Minkowski sum of line intervals (up to a permutation of the summands)?

(e) Same question as (d) but now with Minkowski sums and differences?

Problem 14. Let $e_1, \ldots, e_n$ be the standard coordinate vectors in $\mathbb{R}^n$. For a subset $I \subseteq [n]$, let Δ_I denote the simplex with the vertices e_i for $i \in I$. (Simplices are multidimensional generalizations of triangles.) For example, $\Delta_{\{i\}}$ is just a single point, $\Delta_{\{i,j\}}$ is a line interval between the end-points of two coordinate vectors, $\Delta_{\{i,j,k\}}$ is a triangle whose vertices are the end-points of the coordinate vectors, etc.

(a) Find the number of vertices of the Minkowski sum of the simplices $\Delta_{\{i,j\}}$ for all 2-element subsets $I = \{i, j\} \subset [n]$.

(b) Find the the number of vertices of the Minkowski sum of the simplices Δ_I for all consecutive intervals $I = \{i, i+1, \ldots, j\} \subseteq [n]$.

(c) Find the number of vertices of the Minkowski sum of the simplices Δ_I for all subsets $I \subseteq [n]$.

Problem 15. Let $H_{a,b,c}$ be the hexagon with all angles 120° and side lengths a, b, c, a, b, c (in the clockwise order). Let $N_{a,b,c}$ be the number of tilings of $H_{a,b,c}$ into rhombuses with side lengths 1 (and angles 120° and 60°).

(a) Find a closed formula for $N_{a,b,1}$.

(b) Find a closed formula for $F_n := \sum_{a+b=n} N_{a,b,2}$.

Problem 16. Find a closed formula for the number $N_{a,b,c}$ from the previous problem, for any a, b, c.

Problem 17. Let $T_{n,n+1}$ be the polytope of $n \times (n+1)$ matrices with real non-negative entries, all row sums equal to $n+1$, and all column sums equal to n.

(a) Describe the vertices of the polytope $T_{n,n+1}$.

(b) Find a closed formula for the number of vertices in $T_{n,n+1}$.