PROBLEM SET 4 (due on Thursday 05/05/05)

Problem 1. An *Eulerian tour* in a (directed, underected) graph G is a sequence of its (directed, undirected) edges (i_1, i_2), (i_2, i_3), $\dots$, (i_{N-1}, i_1) such that each edge of G appears once. Find the number of Eulerian tours in the *directed* complete bipartite graph $K_{m,n}$ (each edge appears twice in two different orientations).

Problem 2. Prove the following generalized Hurwitz's identity

$$(x+y)\,(x+y+z_1+\cdots+z_n)^{n-1} = \sum_{I,J} x\,(x+z_{i_1}+\cdots+z_{i_k})^{k-1}\, y\,(x+z_{j_1}+\cdots+z_{j_{n-k}})^{n-k-1}$$

where the sum is over all 2^n subdivisions of the set $[n]$ into a disjoint union of two subsets $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_{n-k}\}$.

Problem 3. For $n \geq 5$, find the number of sequences $(i_1, \dots, i_{5n})$ such that the sequence of (ordered) pairs $(i_1, i_2), (i_2, i_3), \dots, (i_{5n-1}, i_{5n}), (i_{5n}, i_1)$ is a permutation of the set $\{(i,j) \mid i,j \in [n],\ i-j \equiv -2, -1, 0, 1, 2 \pmod n\}$.

Problem 4. Let H_n be the graph formed by the vertices and edges of the n-dimensional hypercube, i.e., the vertices of the graph H_n are binary n-sequences and two sequences are connected by an edge if they differ in one bit. Find the total resistance between the opposite vertices $A = (0, \dots, 0)$ and $B = (1, \dots, 1)$ of H_n, assuming that each edge has the resistance 1 ohm.

Problem 5. Two people A and B play the following game of chance called "walk on the dice edge". There are n cards marked by the numbers $1, \dots, n$. Originally, A has k cards and B has the remaining $n-k$ cards. Each turn the random number generator produces a random integer i uniformly distributed on $[n]$, and the person who has the i-th card passes it to the opponent. The game ends when one person (the winner) collects all n cards. Find the probability that A wins the game.

Problem 6. Fix positive integers n, k, l. A sequence $a_1, \dots, a_n$ of positive integers is called a *generalized parking function* if its weakly increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_i \leq l + k(i-1)$, for $i = 1, \dots, n$. (For $k = l = 1$, these are the usual parking functions.) Find and prove a formula for the number of generalized parking functions.

Problem 7. A *perfect matching* in a graph on $2n$ vertices is a subgraph formed by n pairwise disjoint edges. Let C_n be the n-cycle and [2] be the graph with 2 vertices connected by an edge. Find the number of perfect matchings in the "circular ribbon graph" $C_n \times [2]$.

Problem 8. Find the number of perfect matchings in the torus graph $C_4 \times C_4$.

Problem 9. The 5-dimensional hypercube $\{(x_1, \dots, x_5) \mid 0 \leq x_i \leq 1\} \subset \mathbb{R}^6$ is subdivided by the 4 hyperplanes $x_1 + \cdots + x_5 = k$, $k = 1, 2, 3, 4$ into 5 slices. Find the volumes of all slices. (The sum of the volumes should be equal to $1 =$ volume of the whole hypercube.)

Problem 10. Let us expand the product $\prod_{1 \leq i < j \leq n} (x_i + x_j)$ into a sum of monomials. Show that the number of different monomials in this expansion equals the number of forests (= graphs without cycles) on n labelled vertices. For example, the seven terms of the expansion $(x_1+x_2)(x_1+x_3)(x_2+x_3) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3$ correspond to seven forests on 3 labelled vertices.

Problem 11. Find the number of Eulerian tours in the undirected complete bipartite graph $K_{m,n}$.

Problem 12. Show that any two domino tilings of the $m \times n$ board can be obtained from each other by a sequence of flips (involving 2 dominos forming a 2×2-square). Will an analogues claim be true for an arbitary finite subset of the square grid?

Problem 13. Let H_n be the graph which is the hexagonal region in the infinite triangular lattice on the plane with n vertices on each side of the hexagon. Let M_n be the number of perfect matching in the graph H_n. Calculate the limit $A = \lim_{n\to\infty} \ln(M_n)/n^2$. Write A as an integral and evaluate it up to 4 digits.

Problem 14. The *transportation polytope* $T_{m,n}$ is the set of $m \times n$ matrices (x_{ij}) such that (1) $x_{ij} \geq 0$; (2) $\sum_j x_{ij} = n$, for any $i = 1, \ldots, m$; (3) $\sum_i x_{ij} = m$, for any $j = 1, \ldots, n$. Describe all vertices of the polytope $T_{n+1,n}$. What about vertices of $T_{n+2,n}$?