PROBLEM SET 1 (due on Tuesday 03/01/05)

Problem 1. a) Prove that *queue-sortable* permutations are exactly 321-avoiding permutations. b) Prove that *stack-sortable* permutations are exactly 132-avoiding permutations. c) Find a bijection between these 2 classes of permutations.

Problem 2. Show that the generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ is given by the following continued fraction:

$$\sum_{n=0}^{\infty} C_n x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \dots}}}}.$$

Problem 3. A man stands on the edge of a cliff. He makes 2 steps to the right (away from the cliff) with probability 1/2 or he makes 1 step to the left with probability 1/2. Then he continues to walk in this random fashion. Find the probability that the man dies after making some number of steps.

Problem 4. A *set partition* π of $[n] := \{1, \dots, n\}$ is a way to subdivide $[n]$ into nonempty blocks. A set partition is called *noncrossing* if it contains no two blocks B and B' such that $i, k \in B$ and $j, l \in B'$ for some $i < j < k < l$. Show that the number of noncrossing partitions equals the Catalan number.

Problem 5. a) Suppose that the Schensted correspondence τ sends a permutation w to the pair (P, Q) of standard Young tableaux of the same shape λ. Show that the length of the longest decreasing subsequence in w equals the size of the first column in the Young diagram of shape λ. (In class, we proved a similar statement for the maximal increasing subsequence.)

b) Prove that if $\tau(w) = (P, Q)$ then $\tau(w^{-1}) = (Q, P)$, where w^{-1} denotes the inverse permutation.

Problem 6. Find the number of all permutations w of size 9 such that w contains no increasing subsequences of length 4 and no decreasing subsequences of length 4.

Problem 7. For positive integers $n_1, \dots, n_k$, let $S_{n_1,\dots,n_k}$ be the set of sequences $w = w_1, \dots, w_n$ of the length $n = n_1 + \cdots + n_k$ with n_1 '1's, n_2 '2's,..., n_k 'k's. (Such w's are called *permutations of multisets*.) An *inversion* in $w \in S_{n_1,\dots,n_k}$ is a pair (i, j) such that $1 \le i < j \le n$ and $w_i > w_j$. Let $inv(w)$ be the number of inversions in w. Show that

$$\begin{bmatrix} n \\ n_1, n_2, \dots, n_k \end{bmatrix}_q = \sum_{w \in S_{n_1,\dots,n_k}} q^{inv(w)},$$

where $\begin{bmatrix} n \\ n_1, n_2, \dots, n_k \end{bmatrix}_q := \dfrac{[n]!}{[n_1]! \cdots [n_k]!}$ is the *q-multinomial coefficient.*

Problem 8. Prove the following identity for the q-binomial coefficients

$$\sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

Problem 9. The *length* $\ell(w)$ of a permutation $w \in S_n$ is the minimal number of adjacent transpositions needed to obtain w from the identity permutation. The *number of inversion* is $inv(w) = \#\{(i,j) \mid i < j,\ w(i) > w(j)\}$. Show that the length of a permutation equals its number of inversions: $\ell(w) = inv(w)$.

Problem 10. A *descent* in a permutation $w \in S_n$ is an index i, $1 \leq i \leq n-1$, such that $w_i > w_{i+1}$. The *major index* $maj(w)$ of w is defined as the sum of all descents in w. (For example, $maj(12\dots n) = 0$ and $maj(n\dots 21) = 1 + 2 + \cdots + (n-1)$.) Show that the major index maj is equidistributed with the number of inversions inv, i.e, $\sum_{w\in S_n} q^{maj(w)} = \sum_{w\in S_n} q^{inv(w)}$.

Problem 11. (a) For a permutation $w \in S_n$, let $\tilde{w}$ be the permutation obtained from w by replacing n with 1 and adding 1 to all other entries. Find a relation between major indices $maj(w)$ and $maj(\tilde{w})$ of these permutations. (b) Show that, for any integers i, j, the number of permutations $w \in S_n$ such that $maj(w) \equiv i \pmod{n}$ equals the number of permutation such that $maj(w) \equiv j \pmod{n}$. In other words, the statistics $maj(w) \pmod{n}$ is *uniformly distributed* on S_n. (c) Show that the statistics $inv(w) \pmod{n}$ is also uniformly distributed.

Problem 12. Show that in a *differential poset* (such as the *Young lattice* or the *Fibonacci lattice*) the number of paths of length $2n$ that start and finish at the minimal element equals $(2n-1)!! = (2n-1)(2n-3)(2n-5)\cdots 1$.

Problem 13. Let A and B be two symmetric real $n \times n$-matrices. Assume that A is *positive semi-definite,* i.e., all its eigenvalues are nonnegative. Also assume that B is *positive definite,* i.e., all its eigenvalues are *strictly* positive. Prove that $A + B$ is positive definite.

Problem 14. Let n, k be integers such that $1 \leq k \leq n/2$. Find a bijection π between k-element subsets of $\{1, \dots, n\}$ and $(n-k)$-element subsets of $\{1, \dots, n\}$ that satisfies the property $\pi(I) \supseteq I$.