PROBLEM SET 2 (due on Tuesday, March 12, 2002)

A *trivalent labelled tree* is a tree on the vertices labelled $1, 2, \ldots, n$ such that every vertex has degree 3 or 1 (leaf).

Problem 1 *Find the number of trivalent labelled trees with $n = 2k+1$ vertices.*

Problem 2 *Prove that the number of trivalent labelled trees with $n = 2k$ vertices is given by the formula*

$$\frac{(2k-2)!}{2^{k-1}} \cdot \binom{2k}{k-1}.$$

A *parking function* is a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that, for any $k = 1, \ldots, n$,

$$\#\{i \mid f(i) \leq k\} \geq k.$$

Problem 3 *Prove that the number of parking functions $f : [n] \to [n]$ is equal to $(n+1)^{n-1}$. (You may try to find a bijection between parking functions and trees on $n+1$ labelled vertices.)*

Let $s_i = (i, i+1)$ be the *adjacent transposition* that switches i and $i+1$. The adjacent transpositions s_i, $i = 1, \ldots, n-1$, generate the symmetric group S_n, i.e., every permutation $w \in S_n$ can be written as a product of the s_i. Indeed, every permutation $w(1) \cdots w(n)$ can be obtained from $12 \ldots n$ by a sequence of switching pairs of adjacent entries: like $1234 \to 1243 \to 1423 \to 1432 \to 4132 \to 4312 \to \ldots$. The *length* $\ell(w)$ of w is the minimal possible length of a decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ of w into a product of adjacent transpositions, i.e, the minimal possible number of switches.

Problem 4 *Prove that length $\ell(w)$ of a permutation w is equal to its number of inversions* $\mathrm{INV}(w) = \#\{(i,j) \mid i < j,\ w(i) > w(j)\}$.

Problem 5 *For $w \in S_n$, let*

$$\mathrm{UP}(w) = \#\{i = 1, \ldots, n-1 \mid \ell(s_i w) = \ell(w) + 1\}.$$

Is the statistics UP *equidistributed with any of the statistics (a)* INV *(number of inversions), (b)* DES *(number of descents), (c)* CYC *(number of cycles)?*

The *Bell number* $B(n)$ is the number of partitions of the n-element set $\{1, \ldots, n\}$ into nonempty blocks. For example,

$$B(3) = \#\{(1|2|3), (12|3), (13|2), (23|1), (1|2|3)\} = 5.$$

Problem 6 *Prove that the number of partitions of the $(n+1)$-element set $\{1,\ldots,n+1\}$ into nonempty blocks such that no block contains two adjacent entries i and $i+1$ is equal to the Bell number $B(n)$. For example,*

$$\#\{(1|2|3|4),(13|2|4),(14|2|3),(24|1|3),(13|24)\}=5.$$

The *partition lattice* Π_n is the set of all partitions of the n-element set $\{1,\ldots,n\}$ into nonempty blocks partially ordered by refinement of blocks. In other words, a partition $\pi\in\Pi_n$ is less than or equal than a partition $\sigma\in\Pi_n$ if and only if each block of π is contained in a block of σ. Then the minimal element in Π_n is $\hat{0}=(1|2|\cdots|n)$ and the maximal element is $\hat{1}=(12\cdots n)$. A *maximal chain* in Π_n is an increasing sequence of partitions $\pi_0=\hat{0}<\pi_1<\cdots<\pi_N=\hat{1}$ of maximal possible length.

Problem 7 (bonus) *Find the number of maximal chains in the lattice Π_n.*

An *alternating permutation* is a permutation $w\in S_n$ such that $w(1)<w(2)>w(3)<w(4)>w(5)<\ldots$. Let E_n be the number of alternating permutations in S_n.

Problem 8 *(a) Show that alternating permutations $w\in S_{2k+1}$ are in a bijective correspondence with complete increasing binary trees on $2k+1$ vertices. (A binary tree is complete if each vertex is a leaf or it has both children).*

(b) Show that the numbers E_{2k+1} satisfy the recurrence relation

$$E_{2k+1}=\sum_{i=0}^{k-1}\binom{2k}{2i+1}E_{2i+1}E_{2(k-i)-1}\,,$$

for $k\geq 0$, and $E_1=1$.

(c) Show that the exponential generating function

$$T(x)=\sum_{k\geq 0}E_{2k+1}x^{2k+1}/(2k+1)!$$

satisfies the differential equation

$$T'(x)=1+T(x)^2,\qquad T(0)=0.$$

(d) Prove that

$$\sum_{k\geq 0}E_{2k+1}x^{2k+1}/(2k+1)!=\tan(x).$$

Problem 9 *Show that*

$$\sum_{k\geq 0}E_{2k}x^{2k}/(2k)!=\sec(x).$$

A *path* of length m in the Young lattice $\mathbb{Y}$ from λ to μ is a sequence of Young diagrams $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(m)})$ such that $\lambda^{(0)} = \lambda$, $\lambda^{(m)} = \mu$, and, for each i, the diagram $\lambda^{(i+1)}$ is obtained from $\lambda^{(i)}$ by adding or removing a box.

Problem 10 *Show that the number of paths in* $\mathbb{Y}$ *of length* $2n$ *from* $\lambda^{(0)} = \emptyset$ *to* $\lambda^{(2n)} = \emptyset$ *is equal to* $(2n-1)!! = (2n-1)(2n-3)\ldots 3 \cdot 1$.

Let $J(P)$ be the lattice of order ideals in a poset P.

Problem 11 (bonus) *Show that the lattice* $J(J([2] \times [n]))$ *is unimodular. In other words,*

$$a_0 \leq a_1 \leq \cdots \leq a_r \geq a_{r+1} \geq a_{r+1} \geq \ldots$$

where a_k *is the number of rank* k *elements in* $J(J([2] \times [n]))$.

Let $C_{n,k}$ be the number of permutations $w \in S_n$ that consist of a single n-cycle and such that $w \cdot (12 \cdots n)$ consists of k cycles. Then $\sum_k C_{n,k} = (n-1)!$, $C_{n,n} = 1$, and $C_{n,k} = 0$ when $n - k$ is odd.

Problem 12 (bonus)
(a) Show that $C_{n,n-2} = \binom{n+1}{4}$.
(b) Show that $C_{2k+1,1} = (2k)!/(k+1)$.
(c) Find a formula for $C_{n,k}$.