Problem 1. The hypersimplex $\Delta_{kn}$, for $1 \leq k < n$, is the polytope in $\mathbb{R}^n$ defined as the convex hull of the $\binom{n}{k}$ points $(a_1, \ldots, a_n)$ such that all $a_i \in \{0, 1\}$ and $a_1 + \cdots + a_n = k$. Use the definition of faces of a polytope as supporting faces of a linear function to give an explicit description of
   (a) all edges of $\Delta_{kn}$,
   (b) all facets of $\Delta_{kn}$.

Problem 2. In class, we computed the $f$-vector and $h$-vector of permutohedron and deduced the following identity involving the Stirling numbers of the second kind $S(n, k)$ and the Eulerian numbers $A(n, k)$. (Recall that $S(n, k)$ is the number of set partitions of $[n]$ with $k$ blocks, and $A(n, k)$ is the number of permutations in $S_n$ with $k$ descents.)

$$\sum_{i=0}^{n-1} (n-i)! S(n, n-i) x^i = \sum_{i=0}^{n-1} A(n, i) (x+1)^i.$$

Give a direct combinatorial proof of this identity.

Problem 3. Prove that the normal fan $N_{P+Q}$ of the Minkowski sum $P + Q$ of two polytopes $P$ and $Q$ is the common refinement of the normal fans $N_P$ and $N_Q$.

Problem 4. True or false: Any centrally symmetric 3-dimensional polytope is a zonotope. Prove this claim or find a counterexample (and prove that it is a counterexample).

Problem 5. Prove that each vertex of the Minkowski sum $P + Q$ of two polytopes can be uniquely written as a sum of a vertex of $P$ and a vertex of $Q$.

Problem 6. Find a bijection between integer lattice points of the permutohedron $P_n$ and forests on $n$ labelled vertices.
Problem 7. Prove that the expansion of the product
\[ \prod_{1 \leq i < j \leq n} (x_i + x_j) \]
contains the monomials \( x_1^{a_1} \cdots x_n^{a_n} \) (with nonzero coefficients) for all integer lattice points \( a = (a_1, \ldots, a_n) \) of the (shifted) permutohedron \( P_n + \{(-1, \ldots, -1)\} \). Describe all monomials in this expansion whose coefficients are equal to 1.

Problem 8. Fix \((n-1)(\binom{n}{2})\) nonzero complex constants \( c_{ijk} \), for \( 1 \leq i < j \leq n \) and \( k = 1, \ldots, n-1 \). Assume that the product of any nonempty subset of \( c_{ijk} \)'s is not equal to 1. Consider the following polynomials \( f_k(x_1, \ldots, x_n) \), \( k = 1, \ldots, n-1 \), in the variables \( x_1, \ldots, x_n \):
\[
f_k(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - c_{ijk} x_j).
\]
Find the number of solutions in \((\mathbb{C} \setminus \{0\})^n\) of the system of \( n \) equations in the \( n \) variables \( x_1, \ldots, x_n \) by explicitly solving this system:
\[
\begin{align*}
&f_1(x_1, \ldots, x_n) = 0 \\
f_2(x_1, \ldots, x_n) = 0 \\
&\quad \vdots \\
&f_{n-1}(x_1, \ldots, x_n) = 0 \\
x_n = 1
\end{align*}
\]
Compare your answer with Kushnirenko’s theorem.

Problem 9. For a polytope \( P \) that belongs to the hyperplane \( H := \{(x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n \), we defined the volume \( \text{Vol}_H(P) \) as \( \text{Vol}(p(P)) \), where \( p : \mathbb{R}^n \to \mathbb{R}^{n-1} \) is the projection \( p : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}) \). Also let \( \text{Vol}_{\text{eucl}}(P) \) be the usual \((n-1)\)-dimensional Euclidian volume of \( P \).

For any \( n \), find the constant \( C \) such that \( \text{Vol}_{\text{eucl}}(P) = C \cdot \text{Vol}_H(P) \).

For example, for \( n = 2 \) and the line segment \( P = [(0,0), (1, -1)] \), we have \( \text{Vol}_H(P) = 1 \) and \( \text{Vol}_{\text{eucl}}(P) = \sqrt{2} \), so \( C = \sqrt{2} \).

Problem 10. In class, we constructed a pseudoline arrangement by splitting all triple intersections in the Pappus configuration into 3 double intersections. Give a rigorous proof that this pseudoline arrangement cannot be drawn on the plane with all straight lines.
Problem 11. Let \( G = (V, E) \) be a graph without loops. Pick orientations of all edges in \( G \). Let \( \mathbb{R}^E \) be the vector space of functions \( f : E \to \mathbb{R} \) on edges of \( G \), and let \( \mathbb{Z}^E \subset \mathbb{R}^E \) be the lattice of all integer-valued functions on edges. For a vertex \( v \in V \), \( f_v \in \mathbb{R}^E \) is given by
\[
\begin{cases}
1 & \text{if } e \text{ is an outgoing edge from the vertex } v, \\
-1 & \text{if } e \text{ is an incoming edge to the vertex } v, \\
0 & \text{otherwise}.
\end{cases}
\]
Let \( C_G \simeq \mathbb{R}^m \) be the quotient space of \( \mathbb{R}^E \) by the linear subspace spanned by all \( f_v \), for \( v \in V \). Let \( p : \mathbb{R}^E \to C_G \) be the natural projection to \( C_G \). Also let \( L_G := p(\mathbb{Z}^E) \simeq \mathbb{Z}^m \) be the integer lattice in \( C_G \).

Let \( e_e, e \in E \), denote the coordinate vectors in the space \( \mathbb{R}^E \). (In other words, \( e_e \) is the function on edges of \( G \) which is equal to 1 on the edge \( e \) and 0 on all other edges.)

The cographical vector arrangement is the arrangement of the vectors \( p(e_e) \), for \( e \in E \), in the vector space \( C_G \).

(a) Prove that the cographical vector arrangement is unimodular with respect to the integer lattice \( L_G \).

(b) Describe all bases of the cographical vector arrangement.

Problem 12. Let \( v_1, \ldots, v_N \in \mathbb{Z}^d \) be a unimodular collection of vectors, and let \( Z = \text{Zon}(v_1, \ldots, v_N) \) be the associated zonotope. Prove that the Ehrhart polynomial \( i_Z(t) \) of the zonotope \( Z \) equals
\[
i_Z(t) = \sum_{I \text{ independent subset in } [N]} t^{|I|}.
\]
Use the Ehrhart reciprocity (or some other method) to deduce that the number \( (P_n \setminus \partial P_n) \cap \mathbb{Z}^n \) of integer lattice points in the interior of the permutohedron \( P_n \) equals \((-1)^{n-1}(F_{\text{even}}^n - F_{\text{odd}}^n)\), where \( F_{\text{even}}^n \) (resp., \( F_{\text{odd}}^n \)) is the number of forests on \( n \) labelled vertices with even (resp., odd) number of edges. Can you give a direct proof of this claim?