Problem 1. Let \( f_n(x, y), n \in \mathbb{Z}, \) be the sequence of rational functions in two variables \( x \) and \( y \) given by the initial conditions
\[
f_0 = x, \quad f_1 = y
\]
and the recurrence relation
\[
f_{n+1} = ((f_n)^2 + 1)/f_{n-1}, \quad \text{for } n \in \mathbb{Z}.
\]
Prove that \( f_n(x, y) \) is a Laurent polynomial in \( x \) and \( y \) with positive integer coefficients. Find a combinatorial interpretation of these Laurent polynomials.

Problem 2. (Somos sequences) For a positive integer \( k \), the Somos-\( k \) sequence is the sequence \( (a_0, a_1, a_2, \ldots) \) satisfying the recurrence relation
\[
a_n a_{n-k} = a_{n-1} a_{n-k+1} + a_{n-2} a_{n-k+2} + \cdots + a_{n-[k/2]} a_{n-[k/2]},
\]
with the initial condition \( a_0 = \cdots = a_{k-1} = 1 \).

(a) Prove that the entries of the Somos-4 sequence are integers.

(b) Prove that the entries of the Somos-5 sequence are integers.

Problem 3. (Frieze patterns) A frieze pattern of order \( n \) is an array of numbers \( P = (p_{ij}), i, j \in \mathbb{Z}, i < j < i + n, \) such that
\[
\det \begin{pmatrix} p_{ij} & p_{i+1,j} \\ p_{i,j+1} & p_{i+1,j+1} \end{pmatrix} = 1; \quad p_{i+1} = p_{i+n-1} = 1; \quad p_{ij} > 0.
\]

For a triangulation \( T \) of an \( n \)-gon (with vertices corresponding to elements of \( \mathbb{Z}/n\mathbb{Z} \) clockwise), construct the array \( P(T) = (p_{ij})_{i<j<i+n} \) as follows:

For each \( i \in \mathbb{Z} \), label the vertex \( i \) (mod \( n \)) of \( T \) by 0. Then label by 1 all vertices connected to \( i \) (mod \( n \)) by an edge or a diagonal of \( T \). Then extend this labelling to all vertices of \( T \) using the recursive rule: For any triangle in \( T \) with two vertices labelled by \( a \) and \( b \) and one unlabelled vertex, assign the label \( a + b \) to the third vertex. Then, for \( j \in [i+1, i+n-1] \), let \( p_{ij} \) be the label of the vertex \( j \) (mod \( n \)) in this labelling.

Prove that the map \( T \mapsto P(T) \) is a bijection between triangulations of the \( n \)-gon and integer frieze patterns of order \( n \).
Problem 4. Let $T_0$ be the “star triangulation” of the $n$-gon where the vertex 1 is connected by diagonals with all other vertices. Assign algebraically independent variables $x_e$ to all edges and diagonals $e$ of the triangulation $T_0$.

(a) Construct a $2 \times n$ matrix $A$ (whose entries are rational expressions in the $x_e$) such that, for any edge or diagonal $e = (i, j)$, $1 \leq i < j \leq n$, in the triangulation $T_0$ (i.e., $i = 1$ or $j = i + 1$), $x_e$ is equal to the $2 \times 2$ minor $\Delta_{ij}(A)$ in the columns $i$ and $j$.

(b) Use Plücker relations to deduce that, for any other triangulation $T$ of the $n$-gon, there exists a matrix with the same property. In other words, show that there exists a $2 \times n$ matrix $A_T$ such that $\Delta_{ij}(A_T) = \tilde{x}_e$, for any edge or diagonal $e = (i, j)$ in $T$. Here $\tilde{x}_e$ are algebraically independent variables assigned to all edges and diagonals in $T$.

(c) Now specialize all $\tilde{x}_e$'s to 1. Prove that, for the integer frieze pattern $P(T) = (p_{ij})$ corresponding to the triangulation $T$ (as in previous problem), we have $p_{ij} = \Delta_{ij}(A_T)$, for any $1 \leq i < j \leq n$.

Problem 5. (Diamond Lemma) Let $G$ be a directed graph without infinite directed walks, i.e., any directed walk on $G$ eventually arrives to a sink. Assume that, for any vertex $v$ and two outgoing edges $v \to v'$ and $v \to v''$ in $G$, there exists a vertex $v'''$ with two incoming edges $v' \to v'''$ and $v'' \to v'''$. Prove that any two directed walks on $G$ that start at the same vertex will eventually arrive to the same sink.

Problem 6. (RSK correspondence and the octahedron recurrence) In class, we constructed the map $RSK_n$

$$\{\text{nonnegative } n \times n \text{ matrices}\} \to \{n \times n \text{ reverse plane partitions}\}$$

using the tropical octahedron recurrence and split chessboards. For example, for $n = 2$,

$$RSK_2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \min(b, c) & a + b \\ a + c & \max(b, c) + a + d \end{pmatrix}.$$

Show that this map $RSK_n$ is the same as the one obtained via in the classical RSK insertion algorithm. (Here a pair $(P, Q)$ of semistandard Young tableaux in the image of the classical RSK correspondence is identified (via Gelfand-Tsetlin patterns) with a reverse plane partition of square $n \times n$ shape.)

Problem 7. (Geometric RSK correspondence) Let $X = (x_{ij})$ be an $n \times n$ matrix whose entries are variables $x_{ij}$. For $i, j \in [n]$ and $k \in$
Define \( y_{i,j,k} := \left( \prod_{a \in [i], b \in [j], (a+b \leq k) \text{ or } a+b \geq i+j-k+2} x_{ab} \right) \left( \sum_{P_1, \ldots, P_k} wt(P_1) \cdots wt(P_k) \right) \), where the sum is over all families of \( k \) non-crossing lattice paths \( P_1, \ldots, P_k \) on \( \mathbb{Z} \times \mathbb{Z} \) (with steps \((1, 0)\) and \((0, 1)\)) connecting the points \((i-k+1, j), (i-k+2, j-1), \ldots, (i, j-k+1)\); and the weight of a lattice path \( P \) is the product over its vertices \( wt(P) := \prod_{(a,b) \in P} x_{ab} \).

Also let \( \bar{y}_{ij} := y_{i,j,n+1-\max(i,j)}, \) and let \( z_{ij} = y_{ij}/y_{i+1,j+1}. \) (Assuming that \( y_{ij} = 1 \) if \( i > n \) or \( j > n \).)

Define the geometric RSK correspondence as the birational map \( g_{RSK}^n : (x_{ij})_{i,j \in n} \mapsto (z_{ij})_{i,j \in [n]} \).

(a) The tropicalization \( \text{Trop}(g_{RSK}^n) \) of this birational map is the piecewise-linear map \( \{n \times n \text{ matrices}\} \to \{n \times n \text{ matrices}\} \) obtained from \( g_{RSK}^n \) by replacing the arithmetic operations: “\( \times \)” \( \to \) “\( + \)”, “\( / \)” \( \to \) “\( - \)”, “\( + \)” \( \to \) “\( \max \)”. Show that \( \text{Trop}(g_{RSK}^n) \) is exactly the classical RSK correspondence, i.e., the map \( RSK^n \) from the previous problem.

(b) Let \( \bar{y}_{i,j,k} := y_{i,j,k} / \prod_{a \leq i+k-1, b \leq j+k-1} x_{ab} \). Show that the array \( \bar{y}_{i,j,k} \) satisfies the octahedron recurrence:

\[ \bar{y}_{i,j,k+1} = \bar{y}_{i,j-1,k+1} + \bar{y}_{i-1,j,k+1} + \bar{y}_{i-1,j,k+1} \bar{y}_{i,j,k}. \]

**Problem 8.** (Totally positive and totally nonnegative matrices) Let \( \text{Mat}(m,n) \simeq \mathbb{R}^{mn} \) be the set of all real \( m \times n \) matrices. Prove that the closure (in the usual topology on \( \mathbb{R}^{mn} \)) of the subset of totally positive matrices in \( \text{Mat}(n,n) \) is exactly the set of totally nonnegative matrices. In other words, any totally nonnegative matrix can be approximated by a totally positive matrix.

**Problem 9.** (Minors of upper-triangular matrices) Prove that a generic upper-triangular unipotent \( n \times n \) matrix \( U \) (i.e., \( U \) has 1’s on the main diagonal) has exactly the Catalan number \( C_n := \frac{1}{n+1} \binom{2n}{n} \) of different non-zero minors. For example, the matrix \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) has \( C_2 = 2 \) nonzero minors 1, \( x \); and the matrix \( \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \) has \( C_3 = 5 \) nonzero minors 1, \( x, y, z, xz - y \). Find a bijection between these minors and some of the known combinatorial interpretations of \( C_n \).
Problem 10. (LDU decomposition for totally positive matrices) Let $A$ be an $n \times n$ matrix $A$ with the LDU decomposition $A = LDU$, i.e., $L$ is lower-triangular unipotent, $D$ is diagonal, and $U$ is upper-triangular unipotent. Show that $A$ is a totally positive matrix if and only if each of the factors $L$, $D$, and $U$ is a totally positive lower-triangular/diagonal/upper-triangular matrix. Here we say that $L$ and $U$ are totally positive unipotent lower/upper-triangular matrices if all of their $C_n$ minors that are not identically zero are strictly positive. Likewise, we say that $D$ is a totally positive diagonal matrix if all its diagonal entries are strictly positive.

Problem 11. (Moves of double wiring diagrams) Prove the relations of the form $EE' = AC + BD$ for the minors $\Delta_{I,J}$ of an $n \times n$ matrix that correspond to all moves of Fomin-Zelevinsky’s double wiring diagrams. More explicitly, prove the following relations for minors.

Let $I, J \subset [n]$ such that $|I| = |J|$, let $i', i'', i''', j', j'', j''' \in [n] \setminus I$, and let $j', j'', j''' \in [n] \setminus J$.

(a) $\Delta_{\{i''\} \cup I, \{j'\} \cup J} \Delta_{\{i', i''\} \cup I, \{j', j''\} \cup J} = \Delta_{\{i', i''\} \cup I, \{j', j''\} \cup J} + \Delta_{\{i', i''\} \cup I, \{j', j''\} \cup J} \Delta_{\{i''\} \cup I, \{j'\} \cup J}$.

(This relation corresponds a Coxeter move of 3 strands of the same color in a double wiring diagram.)

(b) **Lewis Carroll identity aka Dodgson condensation:**

$\Delta_{\{i''\} \cup I, \{j'\} \cup J} \Delta_{\{i', i''\} \cup I, \{j', j''\} \cup J} = \Delta_{\{i’, i''\} \cup I, \{j’, j''\} \cup J} + \Delta_{\{i’, i''\} \cup I, \{j', j''\} \cup J} \Delta_{\{i''\} \cup I, \{j'\} \cup J}$.

(This relation corresponds to commuting a “black crossing” through a “red crossing” in a double wiring diagram.)

Problem 12. (Positivity test via solid minors) A minor $\Delta_{I,J}(A)$ of an $n \times n$ matrix $A$ is called solid if $I$ and $J$ are consecutive intervals in $[n]$ and $1 \in I \cup J$. Show that the collection of solid minors $\Delta_{I,J}$ is a positivity test, that is, if all solid minors of $A$ are strictly positive then all minors of $A$ are strictly positive.

Problem 13. Find an example of a positivity test for $n \times n$ matrices that does not come from a double wiring diagram. What is the smallest $n$ for which such an example exists?

Problem 14. (Bruhat decomposition) Let $G = GL_n$ (over some field $\mathbb{F}$), let $B \subset G$ be the Borel subgroup of invertible upper-triangular matrices, and let $W = S_n \subset G$ be subgroup of permutation matrices.
Prove that $G$ has the following disjoint decomposition

$$G = \bigcup_{w \in W} BwB.$$ 

**Problem 15.** (Lusztig’s transformations) (a) Find the “birational subtraction-free bijection” $(x, y, z) \mapsto (x', y', z')$ from $\mathbb{R}_3^3$ to itself such that

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{pmatrix}. $$

(b) Find the bijective map $(x, y) \mapsto (\tilde{x}, \tilde{y})$ from $\mathbb{R}_2^2$ to $\mathbb{R}_2^2$ such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \tilde{y} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & \tilde{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & t_1 \end{pmatrix}$$

for some $t_1, t_2 \in \mathbb{R}_{>0}$.

(c) For a reduced double wiring diagram $D$, the corresponding double Bruhat cell in $GL_n^{TNN}(\mathbb{R})$ is the subset of matrices that can be factorized as

$$X_{i_1}(t_1) \cdots X_{i_l}(t_l) \text{diag}(z_1, \ldots, z_n),$$

where $t_1, \ldots, t_l, z_1, \ldots, z_n \in \mathbb{R}_{>0}$ and each factor $X_{i}(t)$ has the form $I + tE_{i,i+1}$ or $I + tE_{i+1,i}$. (The sequence of these factors is given by black and red crossings in the double wiring diagram $D$.) Show that this double Bruhat cell depends only on the pair of permutations associated with $D$ (and not on a choice of a double wiring diagram).

**Problem 16.** (“Inverse Lindström Lemma”) Show that, for any totally nonnegative $m \times n$ matrix $A$ (not necessarily square), one can find a planar directed acyclic graph $G$ with positive weights on the edges, that can be drawn on the plane so that all its sources $A_1, \ldots, A_m$ are on the left and all its sinks $B_1, \ldots, B_n$ are on the right (both sources and sinks are ordered from bottom to top), such that

$$a_{ij} = \sum_{P: A_i \rightarrow B_j} \text{weight}(P), \quad \text{for any } i, j,$$

where the sum is over directed paths $P$ in $G$ from the source $A_i$ to the sink $B_j$, and the weight $\text{weight}(P)$ of a path $P$ is the product of weights of all its edges.

**Problem 17.** Prove that a point $(p_I)_{I \subseteq \{n\}} \in \mathbb{R}_{\geq n}^{\binom{n}{k}}$ represents a point of the positive Grassmannian $Gr^{>0}(k, n)$ (i.e., the $p_I$ are the Pücker coordinates of a point in $Gr^{>0}(k, n)$) if and only if the $p_I$ satisfy the 3-term Plücker relations.
Problem 18. Prove that a matroid $M$ is a positroid if and only if any minor of $M$ of rank 2 on 4 elements is a positroid.

Problem 19. (a) Calculate the number of $d$-dimensional cells in the totally nonnegative Grassmannian $Gr_{\geq 0}(2,n)$.
   (b) The same question for $Gr_{\geq 0}(3,6)$.

Problem 20. (Strong Bruhat order) Prove the equivalence of the 3 definitions of the strong Bruhat order on $S_n$:
   (1) Covering relations: $u < w$ iff $w = u \cdot (i,j)$ and $\ell(w) = \ell(u) + 1$.
   (2) $u \leq w$ if any reduced decomposition of $w$ has a subword which is a reduced decomposition of $u$.
   (3) $u \leq w$ if some reduced decomposition of $w$ has a subword which is a reduced decomposition of $u$.

Problem 21. (Stembridge’s formula) Assign the weight $(j - i)$ to a covering relation $w < w \cdot (i,j)$, where $i < j$, in the strong Bruhat order on $S_n$. The weight of a saturated chain from the minimal element $id$ to the maximal element $w_0$ is the product of weights of all covering relations in the chain. Prove that the weighted sum over all saturated chains from $id$ to $w_0$ equals $\binom{n}{2}$!.

Problem 22. (Circular Bruhat order) In class, we defined the circular Bruhat order on decorated permutations. Show that it is a ranked poset with the corank function given by the number of alignments of a decorated permutation.

Problem 23. (Skew-symmetrizable matrices) Prove the equivalence of the following 2 definitions:
   (1) An $n \times n$ matrix $B$ is skew-symmetrizable if there exists a diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ such that $DB$ is skew-symmetric.
   (2) An $n \times n$ matrix $B$ is skew-symmetrizable if there exists a diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ such that $BD$ is skew-symmetric.

Problem 24. (Cluster algebras of rank 2) Show that a cluster algebra of rank 2 with exchange matrix $B = \begin{pmatrix} 0 & b \\ -c & c \end{pmatrix}$ is of finite type if and only if $|b \cdot c| \leq 3$.

Problem 25. The diagram $D(B)$ of an exchange matrix $B$ is the directed graph with edges $i \to j$ for $b_{ij} > 0$ weighted by positive integers $|b_{ij}b_{ji}|$. Show that
(a) If $D(B)$ is the cyclically oriented $n$-cycle with weights $1, \ldots, 1$, then $B$ mutation equivalent to an exchange matrix whose Cartan companion is the type $D_n$ Cartan matrix.

(b) If $D(B)$ is the cyclically oriented 3-cycle with edge weights 2, 2, 1, then $B$ is mutation equivalent to an exchange matrix whose Cartan companion is the type $B_3$ Cartan matrix.

(b) If $D(B)$ is the cyclically oriented 4-cycle with edge weights 2, 1, 2, 1, then $B$ is mutation equivalent to an exchange matrix whose Cartan companion is the type $F_4$ Cartan matrix.

Problem 26. (a) Show that the quiver given by any orientation of the $n$-cycle, except the 2 cyclic orientations, is not 2-finite. Thus it is of infinite type.

(b) More generally, if $B$ is any exchange matrix whose diagram is an $n$-cycle (with some edge orientations and some weights), then $B$ is of infinite type in all cases except the 3 cases from the previous problem.

Problem 27. (Associahedron) Let $R = \{e_i - e_j \mid i, j \in [n], i \neq j\} \subset V = \{x_1 + \cdots + x_n = 0\} \cong \mathbb{R}^{n-1}$ be a type $A_{n-1}$ root system. Let $R_{\geq -1} = \{e_i - e_j \mid i < j \text{ or } j = i-1\} \subset R$ be almost positive roots.

Describe all functions $F : \{\text{almost positive roots}\} \to \mathbb{R}$ such that the polytope

$$P_F := \{x \in V \mid (x, \alpha) \leq F(\alpha) \text{ for any } \alpha \in R_{\geq -1}\}$$

is combinatorially equivalent to the associahedron (i.e., $P_F$ has the same normal fan as the associahedron).


Problem 29. (Cyclohedron) Prove the equivalence of the following 3 descriptions of combinatorial structure of the cyclohedron:

1. (symmetric triangulations) The description in terms of centrally-symmetric triangulations of the $(2r+2)$-gon
2. (graph-associahedron) The description as the nested set complex $N_G$ for the graph $G$ equal to the $(r+1)$-cycle.
3. (CFZ-associahedron) The description in terms of pairwise compatible collections of almost positive roots for a type $B_r$ root system.

Problem 30. (Type C cluster algebras) In class, we mentioned that type $B$ cluster algebra can be constructed by folding from a type $A$ cluster algebra. Do a similar construction for a type $C$ cluster algebra.

Problem 31. ($G$-Catalan numbers) For a graph $G$, we defined the $G$-Catalan number as the number of vertices of the corresponding
graph-associahedron $P_G$, i.e., the number of maximal nested sets in the nested set complex $N_G$. Calculate the $G$-Catalan number when $G$ is the Dynkin diagram of type $D_r$.

**Problem 32.** (Haiman’s model for Coxeter-Catalan numbers in type $A$) Let $Q = \{(a_1, \ldots, a_{r+1}) \in \mathbb{Z}^{r+1} \mid a_1 + \cdots + a_{r+1} = 0\}$ (the root lattice), and $h = r + 1$ (the Coxeter number). The Weyl group $W = S_{r+1}$ acts on $Q$ by permutations of the coordinates. Show that the number of $W$-orbits in $Q/(h+1)Q$ equals the Catalan number \( \frac{1}{r+2} \binom{2r+2}{r+1} \).

**Problem 33.** Show that, for type $B_r$, the number of $W$-orbits in $Q/(h+1)Q$ equals the central binomial coefficient $\binom{2r}{r}$.

**Problem 34.** Calculate $\gamma$-vectors for CFZ generalized associahedra.