RSK (cont'd)

Insertion Algorithm

P - a semi-standard Young tableau

j - positive integer

P \leq j: the SSYT w/ 1 more box, obtained as follows:

Algorithm

1) \( r := 1 \) (row index), \( a_r := j \)

2) IF \( a_r \geq \) largest entry of row \( r \),
then add 1 box in the end of row \( r \) filled with \( a_r \),
and STOP.

3) OTHERWISE, find
the smallest entry \( a_{r+1} \) of
row \( r \) s.t. \( a_{r+1} > a_r \).
Replace \( a_{r+1} \) with \( a_r \)

4) Set \( r := r + 1 \)

5) GO TO (2)
\[ P \leq j_1 \]

**Lemma.** If \( j_1 \leq j_2 \)

\[
(P \leq j_1) \leq j_2 = \quad \text{NEW BOX 1}
\]

Then NEW BOX 2 is located immediately to the right, or above of NEW BOX 2.

**Proof**

In each row the entry in the 1st claim (starting with \( j_1 \)) is strictly to the left and less than or equal to the entry in the 2nd claim (starting with \( j_2 \)).

\( \Rightarrow \) The second chain ends before or in the same row as the first chain. \(\square\)
RSK: \( A = (a_{ij}) \leadsto (P, Q) \)

Convert \( A \) into biword

\[
\begin{pmatrix}
  (i_1, j_1) \\
  (i_2, j_2) \\
  \vdots \\
  (i_n, j_n)
\end{pmatrix}
\]

with \( a_{ij} \) pairs \((i, j)\) ordered lexicographically: \((i, j) < (i', j')\) if

\[
\begin{aligned}
i &< i' \quad \text{or} \\
(i = i' \& j < j')
\end{aligned}
\]

insertion tableau:

\( P = ((\varnothing < j_1) < j_2) \leftarrow \ldots \leftarrow j_n \)

recording tableau:

\( Q : \) if at \( k \)-th insertion we added a box to \( P \),

the we add a box to \( Q \) located at the same position, filled with \( i_k \).
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

biword \( w = (1^2)(2^3)(1^3)(2^4)(3^3) \)

\[
P: \emptyset \sim 2 \sim \frac{1}{2} \sim \frac{1}{1} \sim \frac{1}{2} \sim \frac{11}{2} \sim \frac{114}{2} \sim \frac{1113}{24}
\]

\[
P: \emptyset \sim \frac{1}{1} \sim \frac{1}{2} \sim \frac{112}{2} \sim \frac{122}{2} \sim \frac{1122}{23}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \xrightarrow{RSK} \begin{pmatrix}
\frac{1113}{24} & \frac{11}{2} \\
\frac{11}{2} & \frac{122}{23}
\end{pmatrix}
\]

\[
A \xrightarrow{P} Q
\]

weights: \((2,1,1,1)\) \((1,3,1)\)

Column sums of \(A\) \(\cong\) Row sums of \(A\)

Clearly, \(A\) \(\cong\) pair of SSYT's of the same shape

In order to show that this is a bijection, we need to construct inverse procedure.
Inverse Algorithm:

\((P, Q) \rightarrow \text{biword}\rightarrow A\)

1. Find the horizontal strip in \(Q\) filled with largest entries at \(Q\).

2. "Uninsert" entries at \(P\) tableau located in the same positions as in the strip of \(Q\) (found in 1) in the order right to left.

3. Repeat steps 1 & 2 until the tableaux are empty.

Ex.:

\[
P = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 \end{bmatrix}
\]

- The largest entry in previous row which is < 4.

- Need to uninsert entries in these positions from right to left.

- biword \((1, 2, 2, 1, 4, \text{biword})(3)\)

- Matrix \(A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\)
We proved

**Theorem** RSK is a bijection between matrices $A$ & pairs $(P, Q)$ of SSYT's of some shape $\lambda$ st. column/row sums of $A$ are weights of $P$ & $Q$.

This proved Cauchy identity

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda} S_{\lambda}(x) S_{\lambda}(y)$$

How about dual Cauchy?

$$\prod_{i,j} (1+x_i y_j) = \sum_{\lambda} S_{\lambda}(x) S_{\lambda}(y)$$

There is a similar construction, called the dual RSK that gives a bijection:

$$A \longleftrightarrow (P, Q)$$

matrix filled with 0's & 1's of conjugate shapes

shape $(P) = \lambda$
shape $(Q) = \lambda$

column/row sums of $A$ = weights of $P$ & $Q$. 
RSK has many other important properties.

Theorem 1 If $A \overset{RSK}{\rightarrow} (P, Q)$
then $A^T \overset{RSK}{\rightarrow} (Q, P)$.

Can be proved. But not obvious from the classical construction of RSK.

We’ll give another construction where this symmetry is manifest.

Theorem $A \overset{RSK}{\rightarrow} (P, Q)$ of shape $\lambda$.

Then $\lambda_1$ is the length of a "longest weakly increasing" subsequence of $w$.

$\lambda'_1$ is the length of a "longest strictly decreasing" subseq. of $w$. 
Ex. \[ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \] 

\[ (P, Q) \text{ of shape} \]

\[ w = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 \\ 2 & 1 & 1 & 4 & 3 \end{pmatrix} \]

\( \lambda_1 = 3 \)

\( \lambda'_1 = 2 \)

Note: Subsequence does not need to be consecutive.

---

Problem. What is the length of a longest increasing subsequence in a permutation (distribution of the length, etc).

This was one of the reasons why Robinson-Schensted was initially introduced.

---

Let's talk about permutahedron, (Schensted's case)
We have $S_n \hookrightarrow (P, Q)$, a pair of SYTs of same shape $\lambda$, with $|\lambda| = n$.

$w = (w_1, w_2, \ldots, w_n)$ is a permutation.

$\lambda_1$: longest, increasing subseq

$w_{i_1} < w_{i_2} < \ldots < w_{i_{\lambda_1}}$

$i_1 < \ldots < i_{\lambda_1}$

$\lambda'_1$: longest decreasing subseq

$w_{j_1} > w_{j_2} > \ldots > w_{j_{\lambda'_1}}$

$j_1 < j_2 < \ldots < j_{\lambda'_1}$

Symmetry: If $w \hookrightarrow (P, Q)$ then $w^{-1} \hookrightarrow (Q, P)$.
Let \( \mathcal{L}_\lambda \) denote the set of Young tableaux of shape \( \lambda \).

**Corollary**

\[
\sum_{\lambda \vdash n} \mathcal{L}_\lambda = n!
\]

(1) \( \sum \mathcal{L}_\lambda = n! \)

(2) \( \sum \mathcal{L}_\lambda = \# \text{of involutions in } S_n \)

\[\# \{ w \in S_n \mid w^{-1} = w \} \]

**Exercise**

(Will be in P set 1)

\[\sqrt{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!} \]

where \((2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)\).

**Proof of (2)**

\[w \mapsto (\overline{p}, \overline{q}), \quad w^{-1} \mapsto (\overline{q}, \overline{p})\]

\[p = q \iff \text{w is an involution in } S_n\]
Corollary (Erdős–Szekeres Theorem)

Fix \( m, n \geq 1 \). \( N = m \cdot n + 1 \).

Any permutation \( w_1, w_2, \ldots, w_N \) either has an increasing subsequence of length \( m+1 \) or a decreasing subsequence of length \( n+1 \).

Proof. Suppose not.

Longest incr. subseq. has length \( m \) & longest decr. subseq. \( \leq n \).

\[
\begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_k
\end{pmatrix}
\]

\( \lambda_1 \leq m \) & \( \lambda_i \leq n \)

\( \lambda \) fits inside \( m \times n \) rectangle

\( \lambda \) has \( \leq m \cdot n \) boxes.

But we assumed that \( N = |\lambda| = m \cdot n + 1 \). \( \square \)

There is another nice argument for Erdős–Szekeres Thm.

Based on pigeonhole principle.
Greene's Theorem

\[ \lambda_1 \leq \text{increasing subseq.} \]

How about \( \lambda_2, \lambda_3, \ldots \)?

Greene's Theorem:

For any \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \)

Then \( \forall k \)

\[ \lambda_1 + \lambda_k = \max \# \left( \text{union of } k \text{ weakly increasing } \right) \]

Subseq. \( A \subseteq W \)

\[ = \max \{ I_1 \cup \cdots \cup I_k \} \]

each \( I_1, I_k \) is increasing

\[ \lambda_1 + \lambda_k = \max \# \left( \text{union of } k \text{ strictly decreasing } \right) \]

\( D_1 \cup \cdots \cup D_k \) is decreasing

Example:

\[ w = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 8) \]

\[ w = (3 \ 5 \ 8 \ 2 \ 6 \ 4) \]

\[ \lambda_1 = 3 \]

\[ \lambda_1 + \lambda_2 = 6 \]

\[ \lambda_1 + \lambda_2 + \lambda_3 = 7 \]

\[ \lambda' = 4 \]

\[ \lambda' + \lambda_2 = 6 \]

\[ \lambda' + \lambda_2 + \lambda_3 = 8 \]

Warning: \( \Delta + \lambda_k = \max \#(I_1 \cup I_k) \)

\( I_1, I_k \) are increasing

but sometimes it is impossible
to find such \( I_1, I_k \) s.t.

\( I_1 \) is a max. incr. subseq.
How to prove these properties of RSK (symmetry, incr. & decr. subsequences, ...) ?

There is another construction of RSK where these properties become clear.

We want to break Schensted insertion steps into smaller, "more elementary" steps.

This leads to Fomin's growth diagrams (for SYTs & toggle operations (for SSYTs). Kirillov - Berenstein, &...
Let's start with permutations & SYTs.

\[ S_n \xleftrightarrow{\text{Schensted}} \{(P, Q) \mid \text{pair of SYTs at shape } \lambda + n \} \]

Young's lattice \( \mathcal{Y} \)

All Young diagrams ordered by inclusion \( \lambda \subseteq \mu \) if \( \lambda \subseteq \mu \).

Covering relations in \( \mathcal{Y} \)

\[ \lambda \prec \mu \text{ if } \lambda \subseteq \mu \land |\mu \setminus \lambda| = 1 \]
An SYT corresponds to a saturated increasing chain is $\lambda$ from $\emptyset$ to $\emptyset$

Ex. $\emptyset < \square < \square < \square < \square < \square < \square$

\[
\begin{array}{ccc}
1 & 3 \\
2 & 4 \\
5 & \\
\end{array}
\]

---

A pair $(P, Q)$ of SYT's of the same shape $\lambda$ corresponds to a path in $\left( \begin{array}{c} \emptyset < \square < \ldots < \square \lambda > \ldots > \emptyset \end{array} \right)$ from $\emptyset$ to $\emptyset$ with $n$ "up" steps followed by $n$ "down" steps.

Ex.

\[
P = \begin{array}{ccc}
1 & 2 \\
3 & \\
\end{array}
\]

\[
Q = \begin{array}{c}
1 \\
3 \\
2 \\
\end{array}
\]
How about arbitrary paths in $Y$ from $\varnothing$ to $\varnothing$?

**Up & Down operators**

acting on $\mathbb{Z}[Y]$ (the space of formal linear combinations of Young diagrams)

$\mathbb{Z}[Y]$ has linear basis given by $\lambda$s

Some els of $\mathbb{Z}[Y]$:

- $2\varnothing + 25$
- $3 \varnothing + 100 \varnothing$, etc.

$U, D : \mathbb{Z}[Y] \to \mathbb{Z}[Y]$ two linear operators s.t.

$U : \lambda \mapsto \sum \mu$

$D : \lambda \mapsto \sum \mu$

$\mu \in \lambda$

$\mu$ obtained from $\lambda$ by adding a single box.

$\mu \prec \lambda$

$\mu$ is obtained from $\lambda$ by removing a single box.
Ex. \( U : \square \rightarrow \square + \square \)

\( D : \square \rightarrow \varnothing \)

\( U^2(\varnothing) = U(\square) = \square + \square \)

\( U^3(\varnothing) = U(\square + \square) = \)

\[ \square + \square + \square + \square \]

\[ = \square + 2\square + \square \]

etc.

\( U^n(\varnothing) = \sum_{\lambda \in \Lambda} a_\lambda^x \lambda \)

\[ D^3 U^3(\varnothing) = D^3(\square + 2\square + \square) \]

\[ = \varnothing + 2(2 \cdot \varnothing) + \varnothing \]

\[ = 6 \varnothing. \]

How about

\[ D^n U^n(\varnothing) = ? \]
Theorem

$$D^n U^n (\emptyset) = n! \emptyset.$$ 

This is equivalent to

$$\sum_{\lambda \vdash n} \frac{1}{\lambda^2} = n!$$

\[\uparrow\]

There are \(n!\) of paths in \(\nabla\)

from \(\emptyset\) to \(\emptyset\) with \(n\)

“Up” steps followed by

“Down” steps.

We already know this

from RSK.

But is there a simpler

proof?
Lemma \(DU - UD = I\) or \([D, U] = I\).

Proof

\[\begin{array}{c}
\text{up} \\
\Rightarrow M \\
\text{down}
\end{array}
\begin{array}{c}
\text{down} \\
\Rightarrow N \\
\text{up}
\end{array}\]

The coefficient \(d\) in

\[(DU - UD)(\lambda) = \]

# such \(\lambda\)'s - # such \(\mu\)'s

2 cases

\((1) \lambda \neq \nu\)

we want to add a box to \(\lambda \&\) then remove a different box.

we can do these operations in a different order (first remove \(\lambda\) then add)

So \[\exists \mu \text{ such that } (DU - UD)(\lambda) = 0\]

if \(I \neq \lambda\).

Case II \(\lambda = \nu\)

# Add a box to \(\lambda \&\) then remove the same box

# Remove a box from \(\lambda \&\) then add the same box

In \(\lambda\),

\[\begin{array}{c}
\text{k+1 possible } \mu\text{'s}
\end{array}
\begin{array}{c}
\text{k possible } \mu\text{'s}
\end{array}\]

Why?

So \[\exists \mu \text{ such that } (DU - UD)(\lambda) = (k+1) - k = 1\]

we obtain \((DU - UD)(\lambda) = \lambda\).
Claim: The relations

1. \[ \bigwedge U = I \]
2. \( D \emptyset = 0 \)

Imply that \( D^n U^\emptyset(\emptyset) = n! \emptyset \)

Example:

\[
\begin{align*}
DDUU(\emptyset) &= \\
&= (DU DU + DU)(\emptyset) \\
&= (DU UD + DU + UD + I)\emptyset \\
&= (UD + I + I)\emptyset \\
&= 2I
\end{align*}
\]

Each \( D \) is "moving" to the right until it "annihilates" with some \( U \).

There are \( n! \) ways to match all \( n \) \( D \)'s with \( n \) \( U \)'s into "annihilating pairs".
Another argument

$\psi, \phi$ satisfy the same relations as the operators

\[
\begin{align*}
X : \psi(x) \mapsto x \cdot \psi(x) \\
\frac{d}{dx} : \psi(x) \mapsto \psi'(x)
\end{align*}
\]

acting on polynomial ring $\mathbb{Z}[x]$ and $\frac{d}{dx}(1) = 0$.

So the calculation of $D^n \psi^n(0) = ? \psi$ is equivalent to

\[
\left(\frac{d}{dx}\right)^n x^n (1) = \left(\frac{d}{dx}\right)^n (x^n)
\]

\[
= n! \cdot 1 \implies ? = n!
\]

□
This is why Stanley gave the following definition:

**Definition.** A poset $P$ is called a differential poset if

1. $P$ has a unique minimal element $\hat{0}$.

2. The $\text{Up}$ & $\text{down}$ operators acting on $Z[P]$ satisfy

$$\bigcup I \ast \bigcap U J = I.$$ 

**Theorem.** For any differential poset $\mathcal{D}$, $\mathcal{U}^n(\hat{0}) = n! \hat{0}$. 