lost time: 4 formulas for $S_\chi$,

$$S_{\chi} = S_{\chi} = S_{\chi} = S_{\chi}$$

Weyl chcr. formula

How to prove?

- Divided differences operators:
  $$\partial_i : f \mapsto \frac{(1 - S_i)(f)}{x_i - x_{i+1}}$$

- Demazure operators
  $$D_i : f \mapsto \frac{(1 - \frac{x_{i+1}}{x_i} S_i)(f)}{1 - \frac{x_{i+1}}{x_i}}$$

$$\partial_i(\chi_i f)$$
\( D_i \left( x_i^a x_{i+1}^b \right) = x_i^a x_{i+1}^b + x_i^{a-1} x_{i+1}^{b+1} + \ldots \)

\( a > b \)

\( \ldots + x_i^b x_{i+1}^a \)

\( D_i \) commutes with \( x_j \), \( j \neq i, i+1 \)

\( D_i \left( x_j^a \right) = x_j^a D_i \left( x_j^a \right) \).

**Example**  \( n = 3 \),  \( \lambda = (4,2,0) = \begin{pmatrix} 4 & 2 & 0 \end{pmatrix} \)

Let's calculate \( S_{(4,2,0)}(x_1 x_2 x_3) \)

using Demazure operators:

\( S_{\lambda}(x_1 x_2 x_3) = D_1 D_2 D_1 (x^4) \)

We'll represent a monomial \( x_1^a x_2^b x_3^c \) by a point \((a, b, c)\)

in the affine plane \( \mathbb{A}(x, y, z) \mid x + y + z = 6 \) \( \subset \mathbb{R}^3 \)

All monomials will have same degree = 6, so we are "living" in an affine plane \( x + y + z = 6 \).
For example, Kostka number $K_{(1, 2, 0), (2, 2, 2)} = 3$.

Let's check by counting SSYT's:

```
1 2 2
3 3
2 3
1 2 3
1 1 3 3
```
Observation: All non-zero monomials "live" inside a certain polytope (hexagon in this example)

**Def.** The permutohedron

\[ \Pi(\lambda) := \text{conv}(\{(\lambda w_1), \ldots, \lambda w_n) \mid w \in S_n\} \]

is a convex polytope in \( \mathbb{R}^n \).

Fix \( \lambda = (\lambda_1, \ldots, \lambda_n) \).

\[ s_\lambda = \sum_{g \in \mathbb{Z}^n} K_{\lambda g} x^g = \sum_{\mu \text{ partition}} K_{\lambda \mu} m_\mu. \]

Kostka numbers

**Theorem.**

We have \( K_{\lambda \rho} \neq 0 \) if and only if

\[ \rho \in \Pi(\lambda) \cap \mathbb{Z}^n \]

is an integer lattice point of \( \Pi(\lambda) \).

We already mentioned a related result:

**Theorem.** \( K_{\lambda \mu} \neq 0 \) if and only if \( \lambda \geq \mu \) in the dominance order:

\[
\begin{align*}
\lambda_1 &\geq \mu_1, \\
\lambda_1 + \lambda_2 &\geq \mu_1 + \mu_2, \\
\lambda_1 + \lambda_2 + \lambda_3 &\geq \mu_1 + \mu_2 + \mu_3, \\
&\quad \ldots \\
\text{and} \quad |\lambda| &\geq |\mu|.
\end{align*}
\]
The equivalence of these two results follows from:

**Theorem (Rado) Permutohedron**

\[ \Pi(\lambda) = \text{conv} \{ w(\lambda) \mid w \in S_n \} \subset \mathbb{R}^n \]

is given by the following inequalities:

\[ \Pi(\lambda) = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n \mid \begin{array}{c}
\cdot \quad y_{i_1} + \ldots + y_{i_k} \leq \lambda_{i_1} + \ldots + \lambda_{i_k} \\
\text{for any distinct } i_1, \ldots, i_k \\
\cdot \quad y_1 + \ldots + y_n = \lambda_1 + \ldots + \lambda_n \end{array} \right\} \]

In above example, \( \Pi(4,2,0) = \{ (x,y,z) \mid x,y,z \leq 4; \quad x+y, x+z, y+z \leq 6, \quad x+y+z=6 \} \)

Indeed, for \((y_1, \ldots, y_n) \in \mathbb{Z}^n\), Rado’s inequalities \(\leq\)

the weakly decreasing rearrangement \(\mu = (\mu_1, \ldots, \mu_n)\) of \((y_1, \ldots, y_n)\) satisfies \(\mu \leq \lambda\) in the dominance order.
BTW, there is another lesser known linear basis of \( \Lambda \).

\[
b_\lambda = \sum_{\mu \leq \lambda} m_{\mu} \mu = \sum_{\mu \leq \lambda} x^\mu
\]

for a partition \((\lambda_1, \ldots, \lambda_n)\).

i.e. \( b_\lambda \) is obtained from \( s_\lambda \) by replacing all non-zero coeffts. with 1.

**Lemma** \( E b_\lambda | \lambda \) any partition is a linear basis of \( \Lambda \).

**Proof.** Basically, the same argument as for \( s_\lambda \).

\( b_\lambda \) is related to \( E s_\lambda \) by an upper-triangular matrix with 1's on the diagonal. \( \square \)

We have

\[
S_\lambda = \sum_{\mu} A_{\lambda \mu} b_\mu
\]

**Problem:** A combinatorial formula for \( A_{\lambda \mu} \)? Is it true that \( A_{\lambda \mu} \geq 0 \)?

**Example:** \( \lambda = (4, 2, 0) \)

\[
S_{420} = b_{420} + 2b_{221} + b_{222}
\]

Here we are using Schur polynomials \( s(x_1, x_2, x_3) \), i.e. we only keep partitions \( \nu \) with at most 3 parts.

**Theorem.** \( b_{\mu}(x_1, \ldots, x_\mu) = \sum \ w(\frac{x^\mu}{w \in S_n}) \frac{(1-x-y) \cdots (1-x-y)}{(1-y) \cdots (1-y)} \)

Can be deduced from previous formula, which gives \( z \) over lattice points of a polytope.

Compare \( S_\lambda(y_1, \ldots, y_\lambda) \) with \( \sum \ w \frac{X^\lambda}{w \in S_n} \frac{(1-x-y) \cdots (1-x-y)}{(1-y) \cdots (1-y)} \)
Back to 4 formulas for $S_{x}$...

Fix $n$.

$$S_{x}^{\text{sub}} = \partial_{\omega_{x}}(x^{x}) = S_{x}^{\text{den.}} = D_{\omega_{x}}(x^{x})$$

Let $X_{i} : \mathcal{F} \rightarrow x_{i}$ (operator of mult. by $x_{i}$).

$$X^{\alpha} := x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}$$

Then $D_{\xi} = \partial_{\xi} X_{i}$.

---

**Theorem.**

$$D_{\omega_{x}} = \partial_{\omega_{x}} X^{x}$$

---

**Example.** $n = 3$.

$$D_{\omega_{3}} = D_{1} D_{2} D_{1} = \partial_{1} X_{1} \partial_{2} X_{2} \partial_{1} X_{1}$$

$$= \partial_{2} \partial_{1} \omega_{1} X_{1}^{2} X_{2}$$

This is not completely trivial, because $\partial_{1}$ does not commute with $\partial_{2}$.

$$D_{\omega_{x}} = D_{2} D_{1} D_{2} =$$

$$= \partial_{2} X_{2} \partial_{1} X_{1} \partial_{2} X_{2}$$

$$= \partial_{2} \partial_{2} X_{2} X_{1} X_{2}$$
But we can still "move" $X_i$'s through $\sigma_j$'s if we do it smartly: $X_2$ and $\sigma_1$ don't commute

$$\sigma_1 X_1 \sigma_2 X_2 \sigma_1 X_1 = \sigma_1 \sigma_2 X_1 X_2 \sigma_1 X_1$$

We can do this for any $n$, if we pick a reduced decomposition for $w_0$ smartly.

Lemma

$\psi(x_1, x_n) \text{ commutes with } \sigma_i$ if

$$\psi = S_i(\psi).$$

Lemma

$w_0 = (S_1 S_2 ... S_{n-1}) (S_1 S_2 ... S_{n-2}) ... (S_1 S_2) (S_1)$ is a reduced decomposition

For $n=4$:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
S_1 & S_2 & S_1 & S_3 S_2 S_1
\end{array}
\]

a wiring diagram for $w_0$

$(S_1) (S_2 S_1) (S_3 S_2 S_1)$
Proof of $\mathcal{D} w_0 = \partial w_0 x^8$

$\mathcal{D} w_0 = (\partial_1 x_1 \partial_2 x_2 \ldots \partial_{n-1} x_{n-1})$

$(\partial_1 x_1 \partial_2 x_2 \ldots \partial_{n-2} x_{n-2}) \ldots$

$(\partial_1 x_1 \partial_2 x_2) (\partial_1 x_1) = \partial_1 \partial_2 \ldots \partial_{n-1} (x_1 \ldots x_{n-1})$

$\partial_1 \partial_2 \ldots \partial_{n-2} (x_1 \ldots x_{n-2}) \ldots$

$\partial_1 \partial_2 (x_1 x_2) \partial_1 x_1$

$= (\partial_1 \ldots \partial_{n-1}) (\partial_1 \ldots \partial_{n-2}) \ldots (\partial_1 \partial_2) \partial_1$

$\partial_1 x_1 \ldots x_{n-1} (x_1 \ldots x_{n-2}) \ldots (x_1 x_2) x_1$

$= \partial w_0 x^8$. □

So we proved $S^\text{Schub}_x = S^\text{Dem}_x$. 
Theorem. $S_x^\text{class.} = S_x^\text{comb.}

By the fundamental theorem of symmetric functions, elem. functions $e_k$ generate $\Lambda$.
So in order to prove that two linear bases of $\Lambda$, or of $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^S_n$, coincide it is enough to show that they satisfy the same product rule with $e_k$.

**Def:** A skew Young diagram $\lambda/\mu$ is a **vertical $k$-strip** if any row of $\lambda/\mu$ contains at most $k$ boxes, and $|\lambda/\mu| = k$.

Example:

A vertical $7$-strip.
**Pieri Rule (e version)**

\[ \delta_k \cdot S_\lambda = \sum \Delta \mu \]

\( \Delta \mu \) any partition s.t.

\( \mu / \lambda \) is a vertical \( k \)-strip

---

For symmetric polynomials in \( n \) variables, we have

\[ \delta_k(x_1, \ldots, x_n) \cdot S_\lambda(x_1, \ldots, x_n) = \]

\[ = \sum \Delta \mu (x_1, \ldots, x_n) \]

\( \mu \) w/ at most \( n \) parts

\( \mu / \lambda \) is a \( \nu \) end. \( k \)-strip

---

Both versions are equivalent to each other:

\( \Lambda \Rightarrow \Delta_n \) : specialize \( x_{n+1} = x_{n+2} = \ldots = 0 \)

\( \Delta_n \Rightarrow \Lambda \) : take \( n \) sufficiently large
\[ e_1 \cdot S_x = S_x^{class} + S_x^{comb} + S_x \]
\[ e_2 \cdot S_x = S_x^{class} + S_x^{comb} + S_x^{comb} + S_x \]

In order to show that 
\[ S_x^{class} = S_x^{comb} \] is it enough to prove that both 
\[ S_x^{class} \] & \[ S_x^{comb} \] satisfy Pieri rule.

For \[ S_x^{comb} \], Pieri rule will follow from RSK (Robinson-Schensted-Knuth correspondence), which we'll discuss later.

Let's prove Pieri rule for \[ S_x^{class} (x_1, \ldots, x_n) \]
Proof. \( S_x := \frac{\alpha x + s}{\alpha s} \)

\( a_\alpha := \sum_{w \in S_n} (-1)^{e(w)} \ w(x^\alpha) \).

Since \( e_K = e_K(x_1, \ldots, x_n) \) is a symmetric polynomial, we have

\[
e_K \cdot a_\alpha = \sum_{w} (-1)^{e(w)} e_K \ w(x^\alpha)
\]

\[
= \sum_{w} (-1)^{e(w)} \ w(e_K \cdot x^\alpha)
\]

\[
= \sum_{w} (-1)^{e(w)} \ w\left(\sum_{i_1 < \ldots < i_K} x_{i_1} \cdots x_{i_K} \right)
\]

\[
= \sum_{i_1 < \ldots < i_K} a_{x + \mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_K}}
\]

These are the coord. vectors in \( \mathbb{R}^n \)

only the terms where this is a strictly decreasing vector are non-zero.
Thus
\[ e_k S\lambda(x_1, x_n) = \sum_{\mu = \lambda + \delta_{i_1} + \ldots + \delta_{i_k}} S\mu(x_1, \ldots, x_n) \]

where the sum is over \( i_1 < \ldots < i_k \) such that \( \mu \) is a weakly decreasing vector, i.e., \( \mu \) is a valid partition.

This exactly means that the sum is over all \( \mu \)'s obtained from \( \lambda \) by adding a vertical \( k \)-strip.

Ex. 1

```
   1
  / \  \
2   3
  |  |
  4
```

\( i_1 = 1 \), \( i_2 = 2 \), \( i_3 = 3 \), \( i_4 = 5 \).

So we've got Pieri Rule □
We also have a similar Pieri rule for $h_k$

\[ h_k s_\lambda = \sum \limits_{\mu : \mu / \lambda \text{ is a horizontal } k\text{-strip}} s_\mu \]

The 2 versions of Pieri rule are related by the involution \( \omega : \Lambda \to \Lambda \)

\[ \omega : e_k \leftrightarrow h_k \]

Theorem: We have \( \omega(s_\lambda) = s_{\lambda'} \)

\( \lambda' \) is the conjugate partition to \( \lambda \).