last week: \( m_\lambda, e_\lambda, h_\lambda, p_\lambda \)

\( \lambda = (\lambda_1, \ldots, \lambda_e) \) partitions

today: Another basis of \( \Lambda \)
given by Schur symmetric functions \( S_\lambda \).

First, we define Schur polynomials

\[ S_\lambda(x_1, \ldots, x_n) \in \text{finitely many variables } x_1, \ldots, x_n \]
Remark.

Sym. polys vs Sym. functions

Ex. $E_{\chi}(x_1, \ldots, x_n)$ vs $E_{\chi}(x_1, x_2, \ldots)$

$E_{\chi}(x_1, x_2, \ldots) = \lim_{n \to \infty} E_{\chi}(x_1, \ldots, x_n)$

Once a monomial $x_1^{i_1} \cdot x_n^{i_n}$ appears in $E_{\chi}(x_1, \ldots, x_n)$, it will appear in all $E_{\chi}(x_1, \ldots, x_N)$, for $N \geq n$, with the same coeff.

So, if $n \geq 1$, the sym. polynomial $E_{\chi}(x_1, \ldots, x_n)$ contains all info about sym. function $E_{\chi}(x_1, x_2, \ldots)$.
Classical definition of Schur polynomials $S_\lambda(x_1, \ldots, x_n)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$

$$a_\lambda = \det \begin{bmatrix} x_1^{\lambda_1} & x_2^{\lambda_1} & \cdots & x_n^{\lambda_1} \\ x_1^{\lambda_2} & x_2^{\lambda_2} & \cdots & x_n^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{bmatrix}$$

a minor of Vondermonde matrix

$$= \det (x_i^{\lambda_j}) \quad 1 \leq i, j \leq n$$

$$= \sum_{w \in S_n} \text{sign}(w) \times x_1^{\lambda_{w_1}} \times \cdots \times x_n^{\lambda_{w_n}}$$
Clearly

- \( a_x = 0 \) if \( x_i = x_j \) for some \( i \neq j \)
- \( a_x = c \) if \( x_i = x_j \) for some \( i \neq j \)
- \( a_x \) is anti-symmetric with respect to permutations of \( x_1, \ldots, x_n \) and w.r.t. perms of \( x_1, \ldots, x_n \)

\[
a_x (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = -a_x (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)
\]
So WLOG we may assume that $\alpha_1 > \ldots > \alpha_n$ and write $\alpha = \lambda + \delta$

where $\lambda$ is a partition and $\delta = (n-1, n-2, \ldots, 1, 0)$.

$\alpha \delta$ is divisible by

$$\prod_{1 < i < j \leq n} (x_i - x_j) = \delta \delta$$

Vandermonde det.

So $\lambda$ a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$

$$S\lambda(x_1, \ldots, x_n) = \frac{\alpha x + \delta}{\alpha \delta}$$

is a symmetric poly. in $x_1, \ldots, x_n$
Remark \( \alpha \lambda + \delta \) is known as the Weyl character formula for irreducible representations of \( \text{GL}_n \) (type A).

So Schur polynomials are the "characters of irreps of \( \text{GL}_n \)."

In rep. theory, people use notation \( \delta \) instead of \( \delta = (n-1, n-2, \ldots, 1, 0) \).
Ex., \( n = 2 \), \( \lambda = (1, 0) = \mathbb{O} \)

\[ \lambda + \delta = (2, 0) \]

\[ S_0(x_1, x_2) := \begin{vmatrix} x_1^2 & x_2^2 \\ x_0^0 & x_0^0 \\ x_1 & x_2 \\ 1 & 1 \end{vmatrix} \]

\[ = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2 \]

Observation. This is a polynomial with positive integer coeff.
Remark. We allow $\lambda = (\lambda_1, \ldots, \lambda_y)$ to have 0's in the end. One needs to check that

$$a_{\lambda + \delta}$$

does not change if we append 0 to $\lambda$ & subst. $x_{n+1} = 0$.

Lemma

$$a(\lambda_1, \ldots, \lambda_n) + \delta_n = \frac{a(\lambda_1, \ldots, \lambda_n, 0) + \delta_{n+1}}{a_{\delta_{n+1}}}$$

where $\delta_n = (n-1, n-2, \ldots, 0, 0)$. 

Combinatorial def. of $S_\lambda$

**Def.** A semi-standard Young tableau (SSYT) of shape $\lambda$ is a filling of boxes of the Young diagram $\lambda$ by $1, 2, \ldots, n$ s.t. the numbers strictly increase in columns & weakly increase in rows of $\lambda$.

**Ex.**

$$\lambda = (7, 4, 3, 3, 1)$$

$$\beta = (3, 4, 3, 2, 2, 3)$$

$$\begin{array}{cccccc}
1 & 1 & 1 & 2 & 2 & 3 \\
2 & 2 & 3 & 2 & 6 \\
3 & 4 & 4 & 5 & 5 & 6 \\
\end{array}$$

$$\text{shape} \quad \text{weight}$$

$$\gamma = (\beta_1, \ldots, \beta_n)$$

$$\beta_i = \#i's$$

in the tableau.
\[ S_{\lambda}(x_1, \ldots, x_n) = \sum \text{weight}(T) \]

\( T: \text{SSYT} \)

of shape \( \lambda \)

filled \( \text{w/} \ 1, \ldots, n \)

\[ \text{weight}(T) = (\beta_1, \ldots, \beta_n) \]

\[ x \text{weight}(T) = x_1^{\beta_1} \ldots x_n^{\beta_n} \]

**Theorem.** Classical def. of \( S_{\lambda} \) \( \iff \) comb. def of \( S_{\lambda} \)

Since this is a course on combinatorics, we'll use the comb. def. of \( S_{\lambda} \) & prove that \( a \lambda \) & \( a_{\lambda} \) is the same thing
Speaking of Schur symmetric functions...

\[ S_{\lambda}(x_1, x_2, \ldots) = \sum_{T: SSYT \atop \text{of shape } \lambda} x^\text{weight}(T) \]

Clearly, \( S_{\lambda}(x_1, \ldots, x_n) = S_{\lambda}(x_1, \ldots, x_n, 0, 0, \ldots) \)

But it is not immediately clear (from the comb. def.) that \( S_{\lambda} \)

is symmetric.

**Lemma** \( S_{\lambda} \) is a symmetric function.
Proof. Enough to show that

\[ S_{\lambda}(\ldots x_i, x_{i+1}, \ldots) = S_{\lambda}(\ldots x_{i+1}, x_i, \ldots) \]

\[ \forall \ i = 1, 2, \ldots \]

(This implies \( S_{\lambda} \) is invariant w.r.t. any permutation of \( x_i \)'s.

\[ \text{Ex. } S_{\lambda}(x, y, z) = S_{\lambda}(y, x, z) \]

\[ = S_{\lambda}(y, z, x) = S_{\lambda}(z, y, x) \]

Basically, adjacent transpositions generate all permutations.

So we need to show that \( \# \text{SSYT's of shape } \lambda \) and weight \( \beta = (\ldots \beta_i \beta_{i+1} \ldots) \)

\[ = \# \text{SSYT's of shape } \lambda \] and weights \( \tilde{\beta} = (\ldots \beta_{i+1} \beta_i \ldots) \)
Let's construct a bijection

$$\text{SSYT}(\lambda, \beta) \sim \text{SSYT}(\lambda, \tilde{\beta})$$

the set of SSYT's of shape $\lambda$ & weight $\beta$.

$$T \mapsto \tilde{T}$$

a SSYT with $k$ i's $\&$ $e$ i's $\&$ $l$ (ltl)'s $\&$ $k$ (i+1)'s

All boxes of $T$ filled w/ i's form a horizontal $k$-strip
We will only modify boxes filled with i's & (i+1)'s

\[
\begin{array}{cccc}
7 & 7 & i & (i+1) \\
8 & 8 & 8 & 8
\end{array}
\]

\[i = 7\]

\[i+1 = 8\]

We want to modify (only) this part of the tableau and replace it by 14 7's and 6 8's.
We might have some vertical dominoes \( \begin{array}{c} 7 \\ 8 \end{array} \) which we cannot modify. All other boxes come in several rows of the form \( \begin{array}{cccccccc} 7 & \cdots & 7 & 8 & \cdots & 8 \end{array} \).

Replace them by \( \begin{array}{cccccccc} 7 & \cdots & \cdots & 7 & 8 & \cdots & 8 \end{array} \).
If we repeat the operation we go back to the original tableau T. So this is a bijection

\[ SSYT(\lambda, \beta) \leftrightarrow SSYT(\lambda, \beta). \]

This proves that \( Sx \) is symmetric. \( \square \)
Another construction for $S\lambda$

Gelfand–Tsetlin patterns

- triangular arrays with nonnegative integers
- top row is $\lambda_1, \lambda_2, \ldots, \lambda_n$
- adjacent rows are weakly interlaced

\[
\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
M_1 & M_2 & \ldots & M_{n-1} \\
v_1 & v_2 & \ldots & v_{n-2} & y \\
\end{array}
\]
Example. \( n=6 \) \( \lambda=(7,5,5,4,1,0) \)

Gelfond - Tsetlin patterns are in bijection with SSYT's filled with \( 1, \ldots, n \).
Ex. The above GT-patt. corresponds to the tableau

<table>
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<th>1</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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</tbody>
</table>

Lemma. The interlacing condition

\[ \lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n \]

is equivalent to the condition that boxes between \( \lambda \) & \( \mu \) form a horizontal strip.

Horizontal strip \( \lambda/\mu \).
**Corollary** 
\[ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \]

\[ S_\lambda (x_1, \ldots, x_n) = \sum_{P : \text{Gelfand-Tsetlin patterns}} x \]

with top row \( \lambda \)

\[ \text{weight}(P) = (\beta_1, \ldots, \beta_n) \]

\[ \beta_i = (n+1-i)^{th} \text{ row sum of } P \]

\[ - (n-i)^{th} \text{ row sum of } P \]

**Example** 
\( n = 3, \lambda = (2, 2, 0) \)

\[ S_{(2, 2, 0)} (x, y, z) = \sum_{a, b, c \in \mathbb{Z}} x^a y^b z^c q^{a+b-c} r^{4-a-b} \]

**Remark.** We can view such expression for \( S_\lambda \) as a sum over lattice points of a certain polytope \( P(\lambda) \).
Theorem. Schur symmetric functions $s_\lambda$ form a $\mathbb{Z}$-linear basis of $\Lambda$.

Proof. We'll show that Schur $s_\lambda$'s & monomial $m_\mu$'s are related by a triangular matrix

By def. \[
s_\lambda = \sum_{\mu} K_{\lambda \mu} m_\mu
\]

where $K_{\lambda \mu} = \#$ SSYT's of shape $\lambda$ and weight $\mu$. The $K$'s are Kostka numbers.
Def. The dominance partial order (aka majorization order) on partitions of \( n \).

\[ \lambda = (\lambda_1, ..., \lambda_k), \mu = (\mu_1, ..., \mu_l) \]

\( \lambda \geq \mu \) if

- \( |\lambda| = |\mu| \)
- \( \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{l} \mu_i \) \( \forall i \) (assuming \( \lambda_i = \mu_j = 0 \) for \( i > k, j > l \)).
Lemma. The dominance order is generated by the operations $R_{ij}$ on Young diagrams.

$R_{ij} : \lambda \rightarrow \lambda'$

- Move a box in $\lambda$ from its $i$th row to its $j$th row (if the result is a valid Young diagram).
Example. \( n = 4 \)

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & & \\
\cdot & & & \\
\end{array}
\]

(In this case, it is a total order. But in general it is a partial order.)

Theorem. \( K_{\lambda\lambda} = 1 \)

\( K_{\lambda\mu} \neq 0 \) if \( \lambda \preceq \mu \).

Proof of \( K_{\lambda\mu} \neq 0 \Rightarrow \lambda \preceq \mu \).

For any SSYT of shape \( \lambda \) & weight \( \mu \), boxes containing \( 1, 2, \ldots, i \) appear only in first \( i \) rows of \( \lambda \).
So \( M_1 + \cdots + M_i \leq \lambda_1 + \cdots + \lambda_i \) \( \forall i \)

\[ \iff \]

\( \lambda \geq \mu \) in the dominance order.

Thus \( \{ S_{\lambda^2} \} = (K_{\lambda \mu}) \{ \nu_{\mu} \} \)

an upper triangular matrix with 1’s on the diagonal

for any linear extension of the dominance order on partitions of \( \Lambda \).

\[ \Rightarrow S_{\lambda^2} \] is a basis of \( \Lambda \).
For $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ the set $S_\lambda$

$$S_\lambda = \left\{ (i_1, \ldots, i_n) \in \mathbb{Z}^n \mid \text{ monomial } x_1^{i_1} \cdots x_n^{i_n} \text{ occurs in } S_\lambda(x_1, \ldots, x_n) \text{ with non-zero coeff.} \right\}$$

is the set of all integer lattice points in a certain convex polytope $\Pi(\lambda) \subset \mathbb{R}^n$, called the permutohedron

$$\Pi(\lambda) = \text{conv} \left( (\lambda w_1, \ldots, \lambda w_n) \mid w \in S_n \right)$$

Theorem. $S_\lambda = \Pi(\lambda) \cap \mathbb{Z}^n$
The fact $K_{\lambda \mu} \neq 0 \Rightarrow \mu \leq \lambda$ is related to

**Theorem (Rado)**

$$\Pi(\lambda) = \left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n \left| \begin{array}{l}
\sum y_i + \ldots + y_n = 1, \\
y_{w_1} + \ldots + y_{w_i} \leq \lambda_i + \ldots + \lambda_n \\
\forall \ i = 1, \ldots, n \\
\forall \ w \in S_n
\end{array} \right. \right\}$$