More on the hook length formula...

\[ \varphi_\lambda := \# \text{ SYT's of shape } \lambda. \]

Hook length formula: \( \lambda \vdash n \)

\[ \varphi_\lambda = n! / \prod_{a \in \lambda} h(a) \]

We’ve proved it using either

- Hillman–Gross correspondence
- Generalized RSK correspondence

Other ways to prove the hook length formula:

- Gessel–Viennot method \( \Rightarrow \)
  \( \Rightarrow \) determinantal formula for \( \varphi_\lambda \) \( \Rightarrow \)
  \( \Rightarrow \) hook length formula.

- Weyl's dimension formula \( \Rightarrow \)
  \( \Rightarrow \) hook–content formula \( \Rightarrow \)
  \( \Rightarrow \) hook length formula.
Today: A probabilistic proof of the hook length formula, or Hook walk proof due to Greene, Nijenhuis, Wilf, 1979.

- This proof uses hooks & explains why the hook lengths appear in the formula.
- Zeilberger converted this proof into a bijective proof.

Let $H(\lambda) := \prod_{a \in \lambda} h(a)$

**Hook length formula:** \( f_\lambda = \frac{n!}{H(\lambda)} \).

Clearly, \( f_\lambda \) is defined by the recurrence relation:

\[
\begin{align*}
  f_\lambda &= \sum_{\mu, \mu \prec \lambda} f_\mu \\
  f_\lambda &= 1
\end{align*}
\]

For an SYT \( T \) of shape \( \lambda \times n \), let \( \tilde{T} \) be the SYT obtained from \( T \) by removing the box filled with \( n \). Then \( \tilde{T} \) is an SYT of shape \( \mu \prec \lambda \).

\[
T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & \end{bmatrix} \quad \text{and} \quad \tilde{T} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & \end{bmatrix}
\]
In order to prove the hook length formula, it is enough to show that the same recurrence relation holds for the right hand side.

**Need to show:**

$$\frac{n!}{H(n)} = \sum_{\mu < \lambda} \frac{(n-1)!}{H(\mu)}$$

Clearly, \( \frac{0!}{H(0)} = 1 \) — base of recurrence.

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Fix a Young diagram \( \lambda \vdash n \).

The **hook walk** is a random walk on boxes of the Young diagram \( \lambda \) s.t.

- Pick an initial box \( u \in \lambda \) with uniform probability \( \frac{1}{n} \).
- If \( u \) is not a corner box, then jump from box \( u \) to any other box \( u' \neq u \) in the hook of \( u \) with uniform probability \( \frac{1}{h(u)-1} \).
- Then jump from \( u' \) to any other box \( u'' \) in the hook of \( u' \), etc.
- Repeat until we arrive to a corner box \( \delta \).
- Stop.
A hook walk

P(u, S) := the probability that a hook walk starting at box u ends at corner S.

\[ P(u, S) = \sum_{\text{paths}} \frac{1}{h(u)-1} \frac{1}{h(u)-1} \frac{1}{h(u)-1} \ldots \rightarrow S \]

Example

\[ \begin{array}{cc}
  u & S \\
  u & u \\
\end{array} \]

Hook walks:

\[ \begin{array}{c}
  u \rightarrow u' \rightarrow u'' \rightarrow S \\
  u \rightarrow u' \rightarrow u'' \rightarrow S \\
  u \rightarrow u' \rightarrow u'' \rightarrow S \\
  u \rightarrow u' \rightarrow u'' \rightarrow S \\
  u \rightarrow u' \rightarrow u'' \rightarrow S \\
  u \rightarrow u' \rightarrow u'' \rightarrow S \\
\end{array} \]

\[ \begin{array}{c}
  \frac{1}{(6-1)} \frac{1}{(2-1)} \\
  \frac{1}{(6-1)} \frac{1}{(2-1)} \frac{1}{(2-1)} \\
  \frac{1}{(6-1)} \frac{1}{(2-1)} \frac{1}{(2-1)} \\
  \frac{1}{(6-1)} \frac{1}{(2-1)} \frac{1}{(2-1)} \\
  \frac{1}{(6-1)} \frac{1}{(2-1)} \frac{1}{(2-1)} \\
  \frac{1}{(6-1)} \frac{1}{(2-1)} \frac{1}{(2-1)} \frac{1}{(2-1)} \\
\end{array} \]

\[ P(u, S) = \frac{1}{(6-1)} \frac{1}{(2-1)} + \frac{1}{(6-1)} \frac{1}{(2-1)} + \frac{1}{(6-1)} \frac{1}{(2-1)} + \frac{1}{(6-1)} \frac{1}{(2-1)} + \frac{1}{(6-1)} \frac{1}{(2-1)} + \frac{1}{(6-1)} \frac{1}{(2-1)} + \frac{1}{(6-1)} \frac{1}{(2-1)}. \]
Let $P(S) :=$ the probability that a box walk ends at corner box $S$.

Clearly, $P(S) = \sum_{\text{any box } x \neq S} \frac{1}{n} \cdot P(x, S)$

We have

$\sum_{\text{any corner box } S} P(S) = 1$

The sum of all probabilities adds to 1

Example:

\[
P(S_1) = \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}\right) = \frac{1}{3} \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) = \frac{1}{3} \cdot \frac{3}{2} \cdot \frac{4}{3} = \frac{2}{5}.
\]

\[
P(S_2) = \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{1}{2}
\]

$= \frac{1}{5} \left(1 + 1 + \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2}\right) = \frac{1}{5} \left(1 + (1 + \frac{1}{2}) (1 + \frac{1}{3})\right) = \frac{3}{5}.$

\[
\frac{2}{5} + \frac{3}{5} = 1.
\]
Let us calculate these probabilities in a different way...

Some observations:

1. If we fix a corner box $\delta$, then any hook walk ending at $\delta$ belongs to the rectangle.

2. Define the weight of a box in $x$

$$w_t(a) := \frac{1}{h(a)-1}$$

If $w_t(b) = \frac{1}{x}$ and $w_t(c) = \frac{1}{y}$,
then $w_t(a) = \frac{1}{x+y}$. 

$$h(a) + h(d) = h(b) + h(c)$$

$$(h(a)-1) + (h(d)-1) = (h(b)-1) + (h(c)-1)$$
Let us prove a couple of lemmas about paths in a rectangle with arbitrary weights of this form.

Consider the \((k+1) \times (e+1)\) rectangle with boxes weighted, as follows:

\[
\begin{array}{cccc}
\frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_l} \\
\frac{1}{y_1} & \frac{1}{y_2} & \cdots & \frac{1}{y_e} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_l} \\
\end{array}
\]

Here \(x_1, \ldots, x_l, y_1, \ldots, y_e\) are variables.

The box in row \(j\) & column \(i\) has weight \(\frac{1}{x_i+y_j}\) for \(i \in [k]\) and \(j \in [e]\).

Boxes in the last row have weights \(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_k}\).

Boxes in the last column have weights \(\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_e}\).
For any path \( P \),
\[ w(P) := \text{the product of weights of boxes in } P \]

Lemma 1. \[
\sum_{P} w(P) = \frac{1}{x_1 \cdots x_k y_1 \cdots y_e}
\]

- \( P \) is a lattice path from the upper left to the lower right box.
- The sum is over \((k+e)\) lattice points in the rectangle \((k+1) \times (e+1)\) excluding the walk.

Example. \( k = e = 1 \)

\[
\begin{array}{c|c|c|c|c|c}
& 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
x_1 y_1 & x_1 y_1 & x_1 y_1 & x_1 y_1 & x_1 y_1 & x_1 y_1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

\[
= \frac{1}{x_1 \cdot y_1}.
\]

Proof. Induction on \( k + e \).

Base:

Step of induction.

\[
\sum = \sum' + \sum''
\]

The sum over paths starting with a right step.

\[
= \frac{1}{x_1 + y_1} \cdot \frac{1}{x_2 \cdots x_k y_1 \cdots y_e} +
\]

\[
+ \frac{1}{x_1 + y_1} \cdot \frac{1}{x_1 \cdots x_k y_2 \cdots y_e}
\]

\[
= \frac{1}{x_1 \cdots x_k y_1 \cdots y_e}.
\]
Let us now consider the sum over "hook walks" in the rectangle, i.e., paths that can skip over rows or columns that end in its lower right corner.

**Lemma 2.** \[ \sum_{P \text{ a hook walk in (}k+1\text{) x (}k+1\text{) rectangle}} w(P) = \]

\[ (1 + \frac{1}{x_1})(1 + \frac{1}{x_2}) \ldots (1 + \frac{1}{x_k}) \]

\[ (1 + \frac{1}{y_1})(1 + \frac{1}{y_2}) \ldots (1 + \frac{1}{y_l}) \]

**Example.** \( k = l = 1 \)

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
1 & \frac{1}{x_1} & \frac{1}{y_1} & \frac{1}{x_1, y_1} & 1 + \frac{1}{x_1, y_1} & 1 + \frac{1}{x_1, y_1} & 1 + \frac{1}{x_1, y_1} & 1 + \frac{1}{x_1, y_1} & 1 + \frac{1}{x_1, y_1} \\
\frac{1}{x_1} & 1 & \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} \\
\frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} & 1 + \frac{1}{y_1} \\
\end{array}
\]

\[ = \left(1 + \frac{1}{x_1}\right) \left(1 + \frac{1}{y_1}\right). \]

**Proof.** Any "hook walk" is a lattice path for some subrectangle obtained from the \((k+1) \times (k+1)\) rectangle by removing some (possibly empty) subsets of first \( k \) columns and first \( l \) rows.

Each term in the expansion of \((1 + \frac{1}{x_1}) \ldots (1 + \frac{1}{x_k})(1 + \frac{1}{y_1}) \ldots (1 + \frac{1}{y_l})\) corresponds to a choice of such subrectangle. Now Lemma 2 follows from Lemma 1. \( \square \)
Back to hook walks in $\lambda$ with weights of boxes given by \( \frac{1}{h(a)-1} \).

\[
P(S) := \frac{1}{n} \sum_{\pi} P(\pi, S) =\]

\[
\frac{1}{n} \prod_{a \in \text{cohook}(S), a \neq S} \left( 1 + \frac{1}{h(a)-1} \right)
\]

\[
= \frac{1}{n} \prod_{a \in \text{cohook}(S), a \neq S} \frac{h(a)}{h(a)-1}
\]

\[
= \frac{1}{n} \cdot \frac{H(\lambda)}{H(\lambda - \bullet S^*)}
\]

Indeed, all hook lengths in $\lambda$ and $\mu = \lambda - \bullet S^*$ are the same, except the hook lengths of boxes $a \in \text{cohook}(S)$. The latter are decreased by 1 in $\mu$. 

we obtain
\[ \sum_{\text{corner box } \lambda} P(\lambda) = 1 \]

\[ \frac{1}{n} \sum_{\mu < \lambda} \frac{H(\mu)}{H(\lambda)} = 1 \]

\[ \frac{n!}{H(\lambda)} = \sum_{\mu < \lambda} \frac{(n-1)!}{H(\mu)} \]

This is exactly the needed identity for the recurrence in the RHS of the hook length formula. Q.E.D.
Versions of hook length formulas

Shifted shapes

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a \text{strict} partition of \( n \).

The \text{shifted Young diagram} of shape \( \lambda \) is the collection of boxes on the plane such that:
- the \( i \)th row consists of \( \lambda_i \) boxes,
- first boxes in rows are \text{"diagonally justified" as shown below.}

Example

\[ \begin{array}{cccccccc}
1 & 2 & 3 & 5 & 9 & 10 & 15 & 16 \\
4 & 6 & 8 & 12 & 13 & 14 & 17 & 18
\end{array} \]

The shifted Young diagram of shape \( \lambda = (7, 5, 4, 1) \)

Definition: A \textit{standard Young tableau} of shifted shape \( \lambda \) is a filling of boxes of this shifted shape by 1, 2, \ldots, \( n \) (w/o repeated entries) which increase in rows \& columns.

Example

\[ ^{\wedge} \begin{array}{cccccc}
1 & 2 & 3 & 5 & 9 & 10 & 15 \\
4 & 6 & 8 & 12 & 13 & 14 & 17 \\
7 & 11 & 16 & 17 \\
18
\end{array} \]

an SYT of shifted shape
Hooks with a "broken leg" in a shifted shape

\[ \text{hook length } h(a) = \# \text{ boxes in such a hook with a broken leg at } a. \]

\[ \begin{array}{c|c|c|c|c} \ 1 & 2 & 4 & 6 \ 3 & 5 & 3 \ 4 & 2 & 1 & 1 \ \end{array} \]

hook lengths of the shifted shape

**Shifted Hook-length formula**

\[ \text{# SYT's of shifted shape } \lambda = \frac{n!}{\prod_{\alpha \in \lambda} h(\alpha)} \quad (n = \# \text{ boxes in } \lambda) \]

Example \( \lambda = \begin{array}{c|c|c} \ 3 & 2 & 1 \ 2 & 1 \ \end{array} \)

hook lengths

\[ \text{# shifted SYT's } = \frac{5!}{5 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 2. \]

\[ \begin{array}{c|c|c|c} \ 1 & 2 & 4 & 6 \ 2 & 3 \ \end{array} \]

\[ \begin{array}{c|c|c} \ 1 & 2 & 3 \ 4 & 5 \ \end{array} \]

Exercise Prove the shifted hook length formula (i.e., modify the probabilistic "hook walk" proof).
Skew shapes $\lambda/\mu$

In general, $\# SYT$'s of a skew shape $\lambda/\mu$ is not given by a simple product. (This number might involve large prime factors, which are much larger that parts of $\lambda \& \mu$.)

So for a long time people thought that there is no way to generalize the hook length formula to skew shapes.

But Nanay, 2014 proved the following hook length formula for skew shapes,

Let $D$ be any subset of boxes of $\lambda$.

Def. Excited Moves
If $(i,j)\in D$, then we can replace the box $(i,j)$ by $(i,i)$.

Def. For a skew shape $\lambda/\mu$, an excited diagram is any subset of boxes of $\lambda$ obtained from $\lambda/\mu$ by any number of excited moves.
Naruse Hook length formula

Let \( \lambda/\mu \) be a skew shape with \( |\lambda/\mu| = n \) boxes.

\[
\mathfrak{f}_{\lambda/\mu} = n! \left( \sum_{\text{D excitable diagonal for } \lambda/\mu} \frac{1}{\text{hook length of } \lambda/\mu} \right)
\]

Example

\( \lambda/\mu = \begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{array} \)

hook lengths for shape \( \lambda \)

For all long (including the missing boxes in \( \lambda/\mu \))

Exited diagonals:

the original shape

\[
\mathfrak{f}_{2221/11} =
\]

\[
= 5! \left( \frac{1}{3 \cdot 3 \cdot 2 \cdot 1 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 3} \right)
\]

\[
= 9.
\]