Fix \( n \), let \( \lambda=(\lambda_1,...,\lambda_n), \mu=(\mu_1,...,\mu_n), \delta=(\delta_1,...,\delta_n) \) be 3 partitions such that
\[ |\lambda| + |\mu| = |\delta| . \]

**Honeycombs**

are produced

like this,

drawn on the plane.

---

**Saturation Theorem (Knutson-Tao)**

If \( C_{\lambda\mu}^{\delta} \neq 0 \) for some \( \lambda > \delta \),

then \( C_{\lambda\mu}^{\delta} = 0 \).

[KTJ] proved a stronger result:

**Theorem.** For any honeycomb \( H \)

(not necessarily integer) there exists a honeycomb \( \tilde{H} \) with the

same positions of boundary rays

such that all coordinates of all line segments in \( \tilde{H} \) are

integer linear combinations of the coordinates of the boundary rays.

**Why does this theorem imply saturation?**

---

**Proof of (Theorem \( \Rightarrow \) Saturation)**

If \( \lambda > \delta \), then there

exists a honeycomb with

boundary rays \( k_1, -k_{n_1}, k_{n_2}, -k_{n_2}, \ldots, k_{n_\lambda}, -k_{n_\lambda} \).

\[ \Rightarrow \] There exists (not nec. integer)

honeycomb with boundary rays

\( x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_\lambda} \)

(multiply the first honeycomb by \( \frac{1}{2} \))

\[ \Rightarrow \] There exist an integer honeycomb

with boundary rays \( x_1, x_2, y_1, y_2, \ldots, y_{n_\lambda} \)

\[ \Rightarrow \] \( C_{\lambda\mu}^{\delta} \neq 0 \). □
Idea of the proof of the theorem.

Take any honeycomb $H$. If $H$ has a "cycle" $C$, then "deform" the cycle $C$ until one of the edges contracts.

Examples of cycles:

- All angles in $C$ are $120^\circ$.
- All edges in $C$ and also the "third edges" at each vertex of $C$ have lengths $\neq 0$.

Not cycles:

- Angle $= 60^\circ \neq 120^\circ$.
But these are cycles

a “cycle” might have self-intersecting segments of edges lying on top of each other.

Basically, a cycle is any picture “like this” that you draw in the honeycomb.

Lemma: These cycles can “breathe.” More precisely, for a honeycomb $H$ with a cycles $C$, there is a 1-parameter family of deformations of $H$ that change only the coordinates of the lines containing the edges of $C$. 
Examples

We can deform a cycle like this until one of its edges or one of "3rd edges" at some vertex of C contracts.

These deformations don't change the positions of the boundary rays of the honeycomb.

We can keep contracting edges like this until we obtain a honeycomb without cycles.
Lemma. Any honeycomb without cycles is a "forest" i.e., one or more trees drawn on top of each other.

Examples of "trees" and "forests".

Lemma. In a "tree" honeycomb, we can express coord. of all lines as integer linear comb. of the coord. of boundary edges.

For more details and careful definitions, see

Berezin-Berezin polytope

$BZ(\lambda, \mu, \nu) :=$ the polytope in $\mathbb{R}^N$ (where $N = \frac{3 \cdot n^2}{2}$) whose points are $\mathbb{R}$-valued $BZ$-triangles with boundary conditions given by $\lambda, \mu, \nu$.

**Example** $n = 3$

$$BZ(\lambda, \mu, \nu) = \left\{ (x_1, \ldots, x_9) \in \mathbb{R}^9 \mid \begin{align*}
\lambda_1 + x_4 &= \lambda_2, \\
\lambda_2 + x_1 &= \lambda_3 \\
\lambda_3 + x_5 &= \lambda_4, \\
\lambda_4 + x_9 &= \lambda_5 \\
x_6 + x_7 &= \lambda_6, \\
x_6 + x_8 &= \lambda_7, \\
x_3 + x_5 &= x_7 + x_9
\end{align*} \right\}$$

Equivalently, $BZ(\lambda, \mu, \nu)$ is the polytope whose points correspond to honeycombs with boundary rays given by $\lambda, \mu, \nu$. 
Even if $\lambda, \mu, \nu$ are integers, the polytope $BZ(\lambda, \mu, \nu)$ might have non-integer vertices.

**Exercise.** Find 3 partitions $\lambda, \mu, \nu$ such that $BZ(\lambda, \mu, \nu)$ has a non-integer vertex. Present this vertex by a honeycomb or a $BZ$-triangle.

**Theorem (Knutson-Tao).** For integer $\lambda, \mu, \nu$, the polytope $BZ(\lambda, \mu, \nu)$ has at least 1 integer vertex.

The Berenstein-Zelevinsky polytopes include, as a special case, the Gel'fand-Tsetlin polytopes.
Gelfand-Tsetlin polytope
\[ \text{GT}(\lambda, \mu) \subset \mathbb{R}^n \]
\[ (n=2) \]
- the polytope of \( R \)-valued Gelfand-Tsetlin patterns for \( \lambda, \mu \)

\[ \lambda_1, \lambda_2, \mu_1, \mu_2 \]

\[ \nu_1, \nu_2, \mu_1, \mu_2 \]

\[ \lambda_1, \lambda_2, \mu_1, \mu_2 \]

\[ \lambda_1, \lambda_2, \mu_1, \mu_2 \]

The Kostka number
\[ K_{\lambda, \mu} \equiv \# \text{GT}(\lambda, \mu) \cap \mathbb{Z}^n \]

Consider honeycombs which are "very stretched" in one direction

Assume that all \( \delta_i - \delta_m \) are "very large" (compared to the \( \delta_i - \delta_m \)).

\( B(\lambda, \delta, \delta_0 \neq \lambda, \delta_0 \neq \lambda) \)

is given by the inequalities

\[ \text{"edge lengths" } \geq 0 \]

If parts of \( \delta \) are very "spread apart" than the inequalities for edges of 1st direction are automatically satisfied, so we only need to require the inequalities for edges of directions \( \lambda \) and \( \mu \) which are exactly the interlacing conditions for GT patterns.

---

\[ \text{no alone } \]

\[ \text{honeycomb } \]

\[ \text{GT pattern } \]
Corollary \( GT(\lambda, \mu) = \)
\[= BZ(\lambda, \delta, \delta + (\mu_1, \ldots, \mu_i)) \]

and, in particular
\[K \times \mu = C_{\lambda, \delta} \]

the Kostka number

↑ the Littlewood-Richardson coef.

↑ "much larger"

Actually the Gel'fand-Tsetlin polytope \( GT(\lambda, \mu) \) (for integer \( \lambda \) & \( \mu \)) might have non-integer vertices.

Exercise. Find such non-integer vertex of \( GT(\lambda, \mu) \) using honeycombs.
Define the Berenstein-Zelevinsky cone \( BZ(n) \subseteq \mathbb{R}^M \) (\( M = n^2 \)) as the cone of all honeycombs of size \( n \) (without fixing the position of the boundary rays).

\[
(BZ(n, \gamma_i, \lambda)) = \text{the cone of } BZ(n) \text{ by the affine subspace given by fixing the coord. of the boundary rays.}
\]

(Basically, \( BZ(n) \) is a cone of \( BZ \)-triangles of size \( n \) without boundary conditions.)

**Example** \( n = 4 \)

\[
BZ(4) = \mathbb{C}(\gamma_1, \ldots, \gamma_{30}) \subseteq \mathbb{R}^{30} \mid \\
\gamma_1 + \gamma_2 + \gamma_3 = 0, \ldots \\
\gamma_7 \geq \gamma_6 \geq \gamma_1, \ldots \]

Let \( p: \mathbb{R}^M \rightarrow \mathbb{R}^{3n} \) be the projection \( (y_1, \ldots, y_M) \rightarrow \text{coord. of boundary rays} \).

**Ex** For \( n = 4 \)

\[
P: (y_1, \ldots, y_{30}) \rightarrow (y_1, y_4, y_9, \ldots)
\]

only coord. of the boundary rays
The Klyachko cone is the projection of the Berenstein-Zelevinsky cone

\[ \text{Klyachko}(u) = \rho(BZ(C^4)). \]

Equivalently, \((\lambda, \mu, \nu) \in R^3 \) belongs to \( \text{Klyachko}(u) \) if \( u \) is \((R\text{-valued})\) honeycomb with boundary rays given by \( \lambda, \mu, \nu \).

**Saturation Theorem** \( \iff \)

\[ \text{Klyachko}(u) \cap Z^3 \]

\[ = \rho(BZ(C^4) \cap Z^M). \]
Last time we mentioned some inequalities for the Klyachko cone.

For example, for $(\lambda, \mu, \nu) \in \text{Klyachko}(4)$

$$\lambda_1 + \mu_1 \geq \nu_1.$$

In terms of Hermitian matrices $C = A + B$, this means that the largest eigenvalue $\lambda_1$ of $C$ is less than or equal to the sum of the largest eigenvalues $\lambda_1$ and $\mu_1$ of $A$ and $B$.

Let's explain this inequality in terms of honey combs.
We have
\[(a+b+c)(a+b+c) = a+b+c = 0\]

If \(a+b+c < 0\), then \(a+b+c > 0\).

If \(a+b+c < 0\), then these 3 lines are arranged like this:
\[\lambda_1 + \mu_1 \neq \nu_1 \]

This is easy to see by looking at the honeycomb.
Exercise: Prove Wegls inequality using honeycombs, i.e., show that

\[ x_{i+1} + \mu_{i+1} \geq \nu_{i+j+1} \]

for \( i < j < n \)

You need to show that these 3 lines are arranged like this.
Let's us now give all inequalities defining the Klyachko cone.

**Theorem.** The Klyachko cone $\text{Klyachko}(n) \subset \mathbb{R}^{3n}$ is given as follows:

$$\begin{align*}
(\lambda, \mu, \nu) \in \text{Klyachko}(n) \text{ if and only if } \\
\sum_i \lambda_i + \sum_i \mu_i &\geq \sum_i \nu_i \\
\lambda_i &\geq \ldots \geq \lambda_n \\
\mu_i &\geq \ldots \geq \mu_n \\
\nu_i &\geq \ldots \geq \nu_n \\
\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j &\geq \sum_{k \in K} \nu_k
\end{align*}$$

for any triple of subsets $I, J, K \subset \{1, \ldots, n\}$ with $\#I = \#J = \#K = r$, $r \in \mathbb{N}$, such that

the Littlewood–Richardson coefficient $c_{\omega I, \omega J} \neq 0$

where $\omega(I) := (i_1, i_2, \ldots, i_r)$ for $I = \{i_1, \ldots, i_r\}$

the partition associated with subset $I$. 


This gives answers to the following 2 questions:

• If Hermitian $n \times n$ matrices $A + B = C$ with eigenvalues $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)$ $(\nu_1, \ldots, \nu_1)$

  iff $(\lambda, \mu, \nu) \in \text{Klyachko}(n)$

• $C_{\lambda, \mu, \nu} \neq 0$ iff

  $(\lambda, \mu, \nu) \in \text{Klyachko}(n) \cap \mathbb{Z}^{3n}$

Horn conjectured these inequalities for eigenvalues of Hermitian matrices.

Klyachko proved that they are necessary & sufficient.

Knutson–Tao proved that this describes all $C_{\lambda, \mu, \nu} \neq 0$. 
Tom says that non-zero LR-coefficients are described in terms of non-zero LR-coeffs. 

For any partitions $\lambda, \mu, \nu$ with $\leq n$ parts (that can be arbitrarily large) in order to figure out whether $C_{\lambda \mu} \neq 0$ we need to know whether $C_{\lambda \mu} \neq 0$ for 

$$\tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \in \mathbb{R} \times (n-1)$$

for some $r \in \{1, \ldots, n-1\}$.

There are finitely many such $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$'s and all of them have $\leq n$ parts.

So Theorem gives a recursive description of triples of partitions $\lambda, \mu, \nu$ with $C_{\lambda \mu} \neq 0$.

Open Problem. Is there a non-recursive description of such triples of partitions?
An application of honeycombs:

The PRV conjecture

(after Parthasarathy, Ranga Rao, Varadarajan)

Proved by Kumar and Mathieu.

For type $A$.

Theorem Let

$\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_l)$ be two partitions, and $\nu \in S_n$

a weakly decreasing rearrangement of parts of

$(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_n + \mu_l)$.

Then $c_{\lambda \nu}^{\mu} \neq 0$. 
Let’s explain PRV conjecture using honey combs:

Proof. Consider the honey comb obtained by a union of several “Y”s.

This is an integer honey comb. So $\lambda_\mu^1 \neq 0$. □
Special case:

\[ \lambda = \mu = (n, n-1, \ldots, 1) \]

PRV vectors = "sums of 2 permutations"

\[ 1 + w(1), 2 + w(2), \ldots, n + w(n) \]

(rearranged in decreasing order)

for all \( w \in S_n \)

Problem: Give a combinatorial description of PRV vectors.