last time: \( \langle S^\mu \cdot S^\nu, S^\lambda \rangle = \langle S^\lambda, S^\mu \cdot S^\nu \rangle \)

The Littlewood–Richardson coefficients

\[ c^\lambda_{\mu \nu} := \langle S^\mu \cdot S^\nu, S^\lambda \rangle = \langle S^\lambda, S^\mu \cdot S^\nu \rangle \]

\( \lambda, \mu, \nu \) partitions s.t. \(|\lambda| = |\mu| + |\nu| \).

\[ S^\mu \cdot S^\nu = \sum_{\lambda} c^\lambda_{\mu \nu} S^\lambda \]

\[ S^\lambda / S^\mu = \sum_{\nu} c^\lambda_{\mu \nu} S^\nu \]

There are several rep. theoretical/geometric interpretations of \( c^\lambda_{\mu \nu} \):

- In terms of representations of symmetric groups: \( \mu + \nu \), \( \lambda + \mu + \nu \)

\( V^\mu \), \( V^\nu \) irreducible reps. of \( S^\mu \), \( S^\nu \)

(Specht modules)

\[ V^\mu \otimes V^\nu := \text{Ind}_{S^\mu \times S^\nu}^{S^\mu+n} (V^\mu \otimes V^\nu) \]

\[ V^\mu \circ V^\nu := \bigoplus_{\lambda \vdash \mu + \nu} c^\lambda_{\mu \nu} V^\lambda \]

Here \( \lambda, \mu, \nu \) are arbitrary partitions s.t. \( \mu + \nu \), \( \mu + \nu \), \( \lambda \vdash \mu + \nu \).
In terms of $\gamma$, let $G(x) = \sum_{\lambda \vdash n} c^{\lambda\gamma}_{\gamma}$.

For $n \times \lambda \times \gamma = \gamma_{\gamma},$

$p(n, \lambda, \gamma) \leq n \times \lambda \times \gamma$ partitions with at most $n$ parts,

$\lambda(n, \lambda, \gamma) \leq \gamma_{\gamma}$ (parts in $\gamma$ can be repeated)

Since for $\lambda \times \gamma = \gamma_{\gamma}$,

$V(\lambda), V(\gamma), V(\lambda \otimes \gamma)$ irreducible representations of $G(x)$

(‘higher weight modules’)

$V(\lambda \times \gamma) = \bigoplus_{\mu \vdash \gamma} c^{\mu}_{\lambda \times \gamma} V(\mu)$

$\Delta_{\gamma} = \sum_{\lambda \vdash n} c^{\lambda\gamma}_{\gamma} \otimes 1$ (a formal sum of Schur functors $\lambda \times _{\gamma}$)

Remark: These $C_{\gamma}$s are not more general than the LR-coefficients coming from representation of $S_n$.

Set $\lambda \times \gamma$ are usual partitions

(i.e. they hold all nonnegative parts) then $\lambda$ should be a usual partition

(i.e. $\lambda = (a_1, a_2, \ldots)$, $a_i > 0$).

$C_{\lambda \times \gamma} = C_{\gamma}$ if

$\gamma = \lambda \times (a_1, a_2, \ldots)$

Then $\lambda \times \gamma = (a_1, a_2, \ldots)$ for any $a_i > 0$.

$V(\chi_{\lambda \times \gamma}) = V(\lambda) \otimes V(\gamma)$

$V(\gamma) = 1$-dim representation of $G(x)$ given by

$A \mapsto (\text{det}(A))$

for $\gamma \in \mathbb{R}$.

In terms of Schubert varieties in the Grassmannian.

For $n \times k \times \gamma_{\gamma},$

$Gr(n, k) \times \gamma_{\gamma}$ the Grassmannian of $k$-dimensional subspaces of $\gamma_{\gamma}$,

$X_{\lambda \times \gamma_{\gamma}}$ = Schubert varieties in $Gr(n, k)$

$\lambda = (\lambda_1, \lambda_2, \ldots)$, $\lambda_i \geq 0$.

$\nu \in \nu_{\gamma_{\gamma}}$ partitions that fit inside the rectangle $\lambda_{\gamma_{\gamma}}$

$\lambda_{\gamma_{\gamma}}$ the complement of $\lambda$ in the rectangle $\lambda$

$\lambda = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_k - 1, \lambda_{k+1}, \ldots)$

is the Young diagram $\text{dia}(\lambda_{\gamma_{\gamma}})/\lambda$ reduced by 180°.

For $\lambda, \nu \times \gamma_{\gamma} \leq \gamma_{\gamma}$,

$C_{\lambda \times \gamma} = \text{the intersection number of Schubert varieties } X_{\lambda \times \gamma_{\gamma}} \times \nu \times \gamma_{\gamma}$

These numbers $C_{\lambda \times \gamma}$ are related to the LR-coefficients.

$C_{\lambda \times \gamma} = C_{\gamma_{\gamma} \times \lambda}$.
These different representations, theoretical/ geometric interpretations of the LR coefficients, imply the following properties.

**Theorem.**

- **Nonnegativity:**
  
  \( c_{\mu}^{\lambda} \)’s are nonnegative integers.

- **Commutativity (\( S_\mu S_\nu = S_\nu S_\mu \)):**
  
  \[ c_{\mu}^{\lambda} = c_{\nu}^{\lambda} \]

- **Involution \( \omega \):**
  
  \[ c_{\mu}^{\lambda} = c_{\mu'}^{\lambda'} \quad \text{where} \]
  
  \( (\lambda', \mu', \nu') \) are the conjugate partitions.

- **\( S_\kappa \)-symmetry:**
  
  For fixed \( \mu, \nu > 0 \), and \( \lambda \),

  \[ c_{\mu}^{\lambda} \]
  
  is symmetric with respect to any permutation

  \[ A, \lambda, \mu, \nu. \]

  Thus

  \[ c_{\mu}^{\lambda} = c_{\lambda}^{\mu} = c_{\nu}^{\lambda} \]

  \[ c_{\mu}^{\lambda} = c_{\mu}^{\nu} \quad \text{and} \]

  \[ c_{\mu}^{\lambda} = c_{\nu}^{\mu} \quad \text{for} \]

  \( \kappa \neq \mu, \nu \).

So the LR-coefficients satisfy many different kinds of symmetry. Some of these symmetries are clear from one point of view, but mysterious from another point of view.

For example:

\[ S_\mu S_\nu = \sum c_{\mu}^{\lambda} S_\lambda \]

\[ S_{\lambda} = \sum c_{\nu}^{\lambda} S_\nu \]

Why should the coefficient of \( S_\nu \) in \( S_{\lambda} \) be equal to the coefficients of \( S_\nu \) in \( S_{\lambda} \)?

One would like to give a combinatorial formula for the LR-coefficients & explain all these properties combinatorially.
The Littlewood-Richardson Rule
- a combinatorial rule for $C^\lambda_{\mu\nu}$
- "explains" their nonnegativity
- there are several variations of the LR rule related to different ways to think about the LR-coeffs.

The classical LR-rule
Most closely related to the formula:

$$S_{\lambda/\mu} = \sum_{\nu} C^\lambda_{\mu\nu} S_{\nu}.$$ 

We have $S_{\lambda/\mu} = \sum_{\nu} K_{\lambda,\mu,\nu}$
the Kostka number

$$K_{\lambda,\mu,\nu} := \# \{ \text{SSYT's of shape } \lambda/\mu \text{ and weight } \nu \}.$$ 

Hopefully, $C^\lambda_{\mu\nu}$ = the number of some SSYT's of shape $\lambda/\mu$ with weight $\nu$. 
**Definitions**

- A word, i.e., a sequence of positive integers, \( w_1, w_2, \ldots, w_n \), is called a **lattice word** if in any initial subword \( w_1, w_2, \ldots, w_k \), the number of 1's is greater than the number of 2's, which is greater than the number of 3's, and so on. (Lattice words are also called **Yamanouchi words**)

  **Examples:**
  
  1) 1 1 2 1 3 2 3  
  a lattice word
  
  2) 1 1 2 1 3 3 2  
  not a lattice word

- A **Littlewood–Richardson tableau** \( T \) of shape \( \lambda \) and weight \( \mu \) is a semistandard Young tableau such that the word obtained by reading the entries of \( T \) by rows right-to-left top-to-bottom (reverse reading word of \( T \)) is a lattice word.

  Let \( LR(\lambda/\mu, \nu) \) be the set of such Littlewood–Richardson tableaux.

  **Example**

  \[
  \begin{array}{c|cccc}
  & 1 & 1 & 1 & 1 \\
  \hline
  1 & 1 & 2 & 2 & 2 \\
  2 & 2 & 3 & 3 & 3 \\
  3 & 4 & 4 & 4 & 4
  \end{array}
  \]

  a LR-tableau of shape \( \lambda = (19,9,1)/\mu = (6,3,3) \) and weight \( \nu = (9,6,5,3) \) reverse reading word:

  \[
  111122221113333224443111
  \]
Classical LR rule:

Theorem. \[ C^\lambda_{\mu} = \pm \text{LR}(N, \mu, \lambda) \]

the number of Littlewood-Richardson tableaux of shape \( N/N \mu \) and weight \( \lambda \).

History. First stated by Littlewood & Richardson 1934

- Robinson 1938 gave a proof with some gaps.

Example: Suppose \( N = \varnothing \).

Clearly \( S x/\varnothing = S x \)

Let's see why \( \# \text{LR-tableaux} \) of straight shape \( \lambda \)

\[ = \begin{cases} 1 & \text{if } \lambda = \lambda \\ 0 & \text{otherwise} \end{cases} \]

Indeed, for a straight shape \( \lambda \), there is only one LR-tableau:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & \\
3 & 3 & 3 & \\
4 & 4 & & \\
5 & & & \\
\end{array}
\]

\[ \Rightarrow \text{1st row is always filled with 1} \]

\[ \Rightarrow \text{this entry should be 2} \]

\[ \Rightarrow \text{2nd row is filled with all 2's} \]

\[ \text{etc.} \]
Example: \( \lambda = (4, 3, 2), \mu = (2, 1), \) \\
\( \lambda - \mu = (3, 2, 1) \)

LR-table:

\[
\begin{array}{c|c|c}
1 & 1 & \frac{1}{1} \\
\hline
2 & 2 & \\
\hline
3 & 3 & \frac{1}{2}
\end{array}
\]

1 or 2

rev. reading word
11 21 32
rev. reading word
11 22 32

\[
S_\frac{432}{210, 321} = 2.
\]

\[S_{21} \cdot S_{321} = \ldots + 2 S_{432} + \ldots\]

If we fix \( n = 3 \) and think about LR-rule in GL(3) form, we get

\[
C_{210, 321}^{432} = C_{210, 210}^{321}
\]

\[
S_{\frac{432}{210, 321}} \cdot S_{\frac{432}{210, 321}} = S_{\frac{432}{210, 321}} + S_{\frac{432}{210, 321}} + \\
+ S_{\frac{432}{210, 321}} + 2 \cdot S_{\frac{432}{210, 321}} + \\
+ S_{\frac{432}{210, 321}} + S_{\frac{432}{210, 321}} + S_{\frac{432}{210, 321}}
\]
Example: Assume that \( \nu = (k) \)

A LR-tableau of weight \( \nu \) should be a horizontal \( k \)-strip filled with 1's

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

In this case, LR-rule \( \Rightarrow \)
Pieri's rule:

\[
S_k \cdot S_\nu = \sum_{\lambda \text{ s.t. } \lambda/\mu \text{ is a horizontal } k \text{-strip}} S_\lambda
\]

If \( \nu = (k) \), then any LR-tableaux should be a vertical \( k \)-strip filled with 1, 2, ..., \( k \) from to bottom.

\[
S_k \cdot S_\nu = \sum_{\lambda \text{ s.t. } \lambda/\mu \text{ is a vertical } k \text{-strip}} S_\lambda
\]

Remark: In this classical LR-rule, all symmetries that we mentioned (commutativity, inv. \( \omega \), \( S_3 \)-sym.) are pretty non-obvious.
Some variations / generalizations of the LR-rule.

Zelevinsky's pictures

LR-rule is a rule for $\langle Sxy\mu, S\gamma \delta \phi \rangle$.

How about a rule for the inner product of any two skew Schur functions $\langle Sxy\mu, S\gamma \delta \phi \rangle = ?$

Of course, this number can be expressed in terms of the LR-coefficients:

$$Sxy\mu = \sum_{x} c_{\mu|x} S_{x}$$

$$S\gamma \delta \phi = \sum_{x} c_{\gamma \delta|x} S_{x}$$

So $\langle Sxy\mu, S\gamma \delta \phi \rangle =$

$$= \sum_{x} c_{\mu|x} c_{\gamma \delta|x}.$$  

But we would like to give a better rule.
Definition A Zelevinsky picture is a bijection \( \Phi \) between boxes of two skew shapes \( \lambda/\mu \) and \( \lambda/\nu \):

\[ \Phi: \lambda/\mu \rightarrow \lambda/\nu \]

st.

(A) If we fill the shape \( \lambda/\mu \) with \( 1, 2, \ldots, N = \lambda/\mu \)
by rows right-to-left
top-to-bottom (the same order as in the reverse reading),
then the images \( \Phi \) of the
labels under the map \( \Phi \)
form a SYT of shape \( \lambda/\nu \).

(B) The same condition for the inverse map \( \Phi^{-1} \).

Clearly, \# of maps \( \Phi \)
satisfying just the condition (A) 
= \# SYTs of shape \( \lambda/\nu \)
and \# of maps \( \Phi \) satisfying (B) 
= \# SYTs of shape \( \lambda/\mu \).

But the conditions (A) & (B)
together give some new number,
Theorem (Zelevinsky)

\[ \langle S \times \mu, S \rightarrow \lambda \rangle = \# \text{Zelevinsky pictures} \]

\[ f : \lambda / \mu \rightarrow \gamma / \delta. \]

Let's show that this "picture rule" generalizes the classical LR-rule.

Claim: If \( \gamma = \emptyset \), then Zelevinsky's pictures are in bijection with LR-tableaux of shape \( \lambda / \mu \) and weight \( \delta \).

\[ f : \lambda / \mu \rightarrow \gamma / \delta \]

\[ \{ \text{tableau } T \text{ of shape } \lambda / \mu \text{ st. a box } x \text{ of } \lambda / \mu \text{ is filled with } i \text{ if } f(x) \in i^{th} \text{ row of } \delta \}. \]

Check: In this case, the conditions (A) & (B) are equiv. to the definition of LR-tableaux. (Exercise)
Berenstein-Zelevinsky triangles

**Goal:** To reformulate the LR-rule in a more symmetric form (that would explain some symmetries of the LR-coefficients).

**Idea:** Recast the definition of LR-tableaux in terms of Gelfand-Tsetlin patterns.

We will use the "GL(n)-version" of the LR-coefficients.

Fix $n$, $\lambda, \mu, \nu$ are partitions with at most $n$ parts:

- $\lambda/\mu$ has at most $n$ rows
- LR-tableaux are filled with numbers $\in \mathbb{Z}_{1, \ldots, n^2}$.
a LR tableau $T$ of shape $\lambda / \mu$.

$T$ is a SSYT, so it can be written as a Bel'kind-Tsetlin-like pattern of rhombic form:

The LR-conditions (i.e. the lattice word conditions) imply some additional equalities & inequalities for entries of this pattern:

This triangle part of the rhombus is "frozen" because the 1st row of $T$ is filled with $1^3$ or $2^3$.
The "interesting" part of the pattern has triangular form:

\[ \begin{array}{cccc}
H_1 & H_2 & H_3 & \ldots & H_N \\
& \text{Row 1} & \text{Row 2} & \ldots & \text{Row N} \\
\end{array} \]

Reverse reading word:

\[ \begin{array}{cccc}
1 & 2 & 3 & \ldots \\
\text{Row 1} & \text{Row 2} & \text{Row 3} & \ldots \\
L_1 & L_2 & L_3 & \ldots \\
\end{array} \]

Lattice word conditions:

\[
\begin{align*}
\lambda_1 - \mu_1 & \geq \lambda_2 - \mu_2 - a \\
\lambda_2 - \mu_2 - a & \geq \lambda_3 - \mu_3 - b - c \\
\lambda_3 - \mu_3 - b - c & \geq \lambda_4 - \mu_4 - a + c \\
& \text{etc.}
\end{align*}
\]

So

- \# LR tableaux = \#

triangle GT patterns with some additional linear inequalities (coming from lattice conditions)...

In the next lecture we'll show that all these inequalities can be written explicitly in a very symmetric form.