We'll continue with Vershik-Okounkov construction.

Last time: $G$ any finite group

- $G$ has finitely many (up to isomorphism) irreducible representations $V_{\lambda}$, $\lambda \in \mathcal{G}$,
  $|\mathcal{G}| = \# \text{conjugacy classes in } G$

- Group algebra of $G$
  $$C[\mathcal{G}] = \sum_{\lambda \in \mathcal{G}} f_{\lambda} g_{\lambda}, \quad f_{\lambda} \in \mathbb{C}$$
  - algebra of block diagonal matrices (with square blocks)
  - block sizes $d_{1}, \ldots, d_{n}$ are $\dim(V_{\lambda})$
  - (here we assume $\mathcal{G} = \{1, \ldots, N\}$
  - $N = |\mathcal{G}|$)

Explicitly: Pick linear bases in all $V_{\lambda}$'s. Then the representation $V_{\lambda}$ is given by homomorphism:

$$R_{\lambda}: G \to \text{GL}_{d_{\lambda}}$$

$g \mapsto R(g) = \begin{bmatrix} R_{1}(g) & & \\ & \ddots & \\ & & R_{n}(g) \end{bmatrix}$

This linearly extends to the map $C[\mathcal{G}] \to \sum$ block diagonal matrices

$$\sum_{\lambda \in \mathcal{G}} f_{\lambda} g_{\lambda} \mapsto \sum_{\lambda \in \mathcal{G}} f_{\lambda} R(g)$$
The center of the group algebra

\[ Z_{C[G]} := \left\{ f \in C[G] \mid f g = g f \quad \forall g \in G \right\} \]

\[ = C_{\text{const}}(G) := \left\{ \sum_{g \in G} f_g g \mid f_g \text{ is a class function on } G \right\} \]

\[ \alpha_1, \ldots, \alpha_n \in C \]

\[ G_0 \subset G_1 \subset G_2 \subset \ldots \]

any sequence of included groups.

- Brattelli diagram: directed graph on vertex set \( \bigcup_{i \geq 0} G_i \)

edges correspond to "branching rule" \( \text{Res}_{G_n}^{G_{n-1}} V_\lambda = \bigoplus V_\mu \)

\[ m \overset{\mu}{\longrightarrow} \lambda \quad \text{if } V_\mu \text{ has appearance } \]

\[ G_n - G_{n-1} \text{ in } \text{Res}_{G_n}^{G_{n-1}} V_\lambda \]

with multiplicity \( m \).
- \( \dim V_\lambda = \# \text{ directed paths} \)

\[
T = (\emptyset \to \ldots \to \lambda)
\]

in the Bratteli diagram.

(Here \( G_0^\wedge = \{ \emptyset \} \))

- Gelfand-Tsetlin basis of \( V_\lambda \)

\[
\left\{ \mathcal{S}_T \right\} \mid T = (\emptyset \to \ldots \to \lambda)^3
\]

\[
V_\lambda \xrightarrow{\text{Res}} \bigoplus_{\mu \in G_n^\wedge} V_\mu \xrightarrow{\text{Res}} \bigoplus_{\nu \in G_{n-1}^\wedge} \bigoplus_{\nu \in G_{n-2}^\wedge} \bigoplus \bigoplus V_0
\]

\( \sim \ldots \sim \)

\[
V_\lambda = \bigoplus \left\{ \text{one-dimensional spaces} \right\}
\]

Then pick a generator \( \mathcal{S}_T \) in each 1-dim space.

- If Bratteli diagram does not have multiple edges, then a Gelfand-Tsetlin basis is unique up to rescaling the vectors \( \mathcal{S}_T \).
$G_0 \subset G_1 \subset G_2 \subset \ldots$

$[G_0] \subset [G_1] \subset [G_2] \subset \ldots$

$Z_0 \quad U \quad U \quad \ldots$

$Z_n : = ZC[G_n] \quad \text{the center of group } G_n$

- **Gelfand-Tsetlin subalgebra**

$GT_n \subset [G_n]$

$GT_n = \text{algebra generated by } Z_0, Z_1, Z_2, \ldots$

**Proposition** If we pick a GT-basis in each $V_\lambda$. Then

$GT_n \cong \{ \text{all diagonal matrices} \} \cong \begin{bmatrix} 
\begin{array}{ccc}
1 & & \\
& 2 & \\
& & 3
\end{array}
\end{bmatrix}
$

$Z_n$

In particular, $GT_n$ is a maximal commutative subalgebra of $CLG_n$. 
The centralizer of $GL_n$ in $GL_{n+1}$ is $Z_{n+1} := \{ g \in GL_{n+1} \mid \forall h \in GL_n \text{ all elements of } Z_{n+1} \text{ that commute with } g \} = GL_n \backslash GL_{n+1}$.

**Proposition** The Bratteli diagram does not have multiple edges if all centralizers $Z_{n+1}$ are commutative.

In this case:
- GT-basis $\mathcal{G}_{T_3}$ is unique (up to rescaling at $T_3$)
- A vector $S \in V_\chi$ belongs to GT-basis (up to rescaling) iff $S$ is a common eigenvector of all elements of $GT_n$
- Basis elements in GT-basis are uniquely determined by eigenvalues of elements of $GT_n$

We will not prove above Proposition. Proofs can be found in the paper by Vershik - Okounkov. They are not very hard. This is basically linear algebra.
Let's now specialize the above general construction to the case of symmetric groups $S_n$ included into each other in the standard way:

$$S_0 < S_1 < S_2 < S_3 < \ldots$$

**Young - Jucys - Murphy elements:** For $i=1,2,\ldots$

$$X_i := (1, i) + (2, i) + \ldots + (i-1, i)$$

1. $X_1 = 0$
2. $X_2 = (1, 2)$
3. $X_3 = (1, 3) + (2, 3)$
4. $X_4 = (1, 4) + (2, 4) + (3, 4)$

etc.

Recall, $Z_n := Z[C[S_n]]$

$$Z_{n-1, 1} = \text{centralizer of } Z[C[S_{n-1}]] \text{ in } Z[C[S_n]]$$

$GT_n = \text{subalg. of } C[S_n]$ generated by $Z_1, Z_2, \ldots Z_n$
$Z_n \subset Z_{n-1} \subset GT_n \subset C[S_n]$

Clearly, $X_n = \mathbb{Z}$ all transp in $S_n$
- $\mathbb{Z}$ all transp in $S_{n-1}$
$\in \langle Z_n, Z_{n-1} \rangle \subset GT_n$.

**Theorem.** $GT_n$ is the algebra generated by $X_1, X_2, \ldots, X_n$.

$$GT_n = \langle X_1, X_2, \ldots, X_n \rangle.$$  

This follows from.

**Theorem.** $Z_{n+1}$ is generated by $Z_{n-1}$ and $X_n$.

$$Z_{n-1,1} = \langle Z_{n-1}, X_n \rangle.$$  

In particular, $Z_{n-1,1}$ is commutative. Thus the Bratteli diagram does not have multiple edges.

(It is easy to see that $X_n$ commutes with $Z_{n-1}$.)

This claim can be proved by induction.
For a partition \((C_1, \ldots, C_e)\)
\[ n = C_1 + \ldots + C_e \]

Let 
\[ [C_1, \ldots, C_e] := \sum_{w \in \mathbb{C}[S_n]} w \] 
with cycle type \((C_1, \ldots, C_e)\)
\[ w = (\ldots) (\ldots) \ldots (\ldots) \]
\[ C_1 \quad C_2 \quad \ldots \quad C_e \]

A **marked partition** is
a partition with one marked part
\[ n = C_1 + \ldots + C_e \]  
\((C_1)\) non necessarily the largest part

\[ [\overline{C_1}, \ldots, \overline{C_e}] := \sum_{w \in \mathbb{C}[S_n]} w \] 
with cycle type \((C_1, \ldots, C_e)\)
such cycle of size \(C_i\)
contains \(n\).

**Example**
\[ [\overline{2}, 1, \ldots, 1] = \] 
\[ = \text{the sum of all transpositions} \] 
in \(S_n\).

\[ [\overline{2}, 1, \ldots, 1] = \text{the sum of all transp. } (i, n) \text{ in } S_n \]
\[ = i \times n. \]
Easy lemma:

1. $\mathbb{Z}_n$ has linear basis $[c_1, \ldots, c_d]$ (given by all partitions of $n$)
2. $\mathbb{Z}_{n+1}$ has linear basis $[c_1, \ldots, c_d]$ (given by all marked partitions of $n$)

(This directly follows from definitions of $\mathbb{Z}_n$, $\mathbb{Z}_{n+1}$)

All above claims about JM-ets follow from

**Lemma.** All $[c_1, \ldots, c_d]$ ($c_1 + \ldots + c_d = n$), can be expressed in terms of $[\mathbb{Z}^*, c_1, c_2, \ldots, c_d]$ ($c_1 + \ldots + c_d = n-1$) and $X_n$.

**Proof.** $X_n \cdot [\mathbb{Z}^*, c_1, c_2, \ldots, c_d] = ?$

2 cases: $i \in \text{marked cycles}$, $i \in \text{some other cycles}$ $c_j$

$X_n [\mathbb{Z}^*, c_1, \ldots, c_d] = \sum_{c_1 + \cdots + c_d = n} (\text{non-zero coeff}) [\mathbb{Z}^*, c_1, c_2, \ldots, c_d]$

$= \sum_{i \in \mathbb{Z}^*, c_3} (\text{non-zero coeff}) [\mathbb{Z}^*, c_1 + c_3, c_2, \ldots, c_d]$

One can use this identity to prove the lemma by induction.

**Exercise.** Do this.
Lemma \( \Rightarrow \) Each \( [c_1, \ldots, c_n] \) \( (c_1 + \cdots + c_n = n) \) can be expressed in terms of \( x_1, x_2, \ldots, x_n \).

\[ \Rightarrow \] each element in \( \mathbb{Z}_n \)

\[ \Rightarrow \] can be expressed in \( x_1, \ldots, x_n \).

\[ \Rightarrow \] \( G \mathbb{T}_n = \langle x_1, \ldots, x_n \rangle \).

**Examples**

\( \left[\begin{array}{c}2 \\ n-2\end{array}\right] = x_1 + \cdots + x_n \)

\( \left[\begin{array}{c}3 \\ n-3\end{array}\right] = x_1^2 + \cdots + x_n^2 - \binom{n}{2} \).

etc.

Now we have:

**Corollary** Each irreducible representation \( V_\chi \) of \( S_n \) has a unique (up to rescaling) basis \( \{ G \mathbb{T}_n \} \) (GT-basis)

such that the vectors \( u_\lambda \) are the common eigenvectors of JM-elements \( x_1, \ldots, x_n \).

Each basis vector \( u_\lambda \) is uniquely determined by the collection \( (\lambda_1, \ldots, \lambda_n) \) of eigenvalues

\[ x_i u_\lambda = \lambda_i u_\lambda. \]
Elements $\mathbf{T}$ of GT-basis can be labelled by
- paths $T$ in the Bravetti diagram,
- vectors $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$
  of eigenvalues of $X_1, \ldots, X_n$.

$T \leftrightarrow (\alpha_1, \ldots, \alpha_n)$

Let $\text{Spec}(n) :=$ the set of all possible vectors $(\alpha_1, \ldots, \alpha_n)$ for all irreps of $S_n$.

Equiv. relation

$(\alpha_1, \ldots, \alpha_n) \equiv (\alpha'_1, \ldots, \alpha'_n)$ if

$(\alpha_1, \ldots, \alpha_n)$ and $(\alpha'_1, \ldots, \alpha'_n)$ correspond to vectors $\mathbf{U}_T$, $\mathbf{U}'_T$ in the GT-basis of the same irreducible representation $V_x$.

Our goal is to describe $\text{Spec}(n)/\equiv$ combinatorially.
Our main tool is

**Theorem** The elements $s_1, \ldots, s_{n-1}$ and $x_1, \ldots, x_n$ in $\mathcal{U}[S]$ satisfy the relations:

- Coxeterrels. for $s_1, \ldots, s_{n-1}$
- $x_i x_j = x_j x_i, \forall i, j$
- $s_i x_j = x_j s_i$, if $j \neq i+1$
- $s_i x_i = x_{i+1} s_i - 1$
- $s_i x_{i+1} = x_i s_{i+1}$.

**Proof** Easy verification.

**Def.** The algebra with above relations is called

Degenerate Affine Hecke Algebra (DAHA).
Local Analysis of Spec(\(I_n\)).

\((\alpha_1, \ldots, \alpha_n) \in \text{Spec}(I_n)\)

corresponds to basis vector \(s = \alpha_1 \mathbf{v}_1\) in \(\mathbf{V}_\lambda\)

- \(\alpha_1 = 0\) (because \(\lambda_1 = 0\))

Suppose \(\alpha_i = a \neq 0\), \(\alpha_{i+1} = b \neq 0\)

\[X_i s = a \mathbf{v},\]

\[X_{i+1} s = b \mathbf{v}\]

Let \(s' = s_{i+1}(s) \in \mathbf{V}_\lambda\)

Consider 2 cases:

I. \(s \propto s'\) are linearly dependent

\[s_{i+1}^2 = 1 \implies s' = \pm s.

Data: \(s_i X i + 1 = s_i X_{i+1}\)

apply this to eigenvector \(s\).

\[\pm a s + s = \pm b s\]

\[b = a + 1\]
If $S, S'$ are linearly independent $\Rightarrow$ they span a 2-dim subspace in $V_x$.

The operators $X_i, X_{i+1}, S_i$ act on the subspace $\langle S, S' \rangle$ by matrices:

$$X_i = \begin{pmatrix} a & -1 \\ b & \ 1 \end{pmatrix}, \quad X_{i+1} = \begin{pmatrix} 0 & a \\ b & 1 \end{pmatrix}, \quad S_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X_i S = a_i S$$

$$X_i S' = X_i S_i(S) = (S_i X_{i+1} - 1) S$$

$$= -S + bS'$$

etc.

**Observation** $a \neq b$

(Otherwise $X_i$ has a non-trivial Jordan block)

$$\begin{pmatrix} a & -1 \\ 0 & a \end{pmatrix}$$

but we know that $X_i$ is diagonalizable.
Let's use the basis $\hat{5}, \tilde{5}$ instead of $\{\delta, \tilde{5}\}$ where $\tilde{5} = 5 + (b-a)\delta$.

$X_i: \tilde{5} \rightarrow b\tilde{5}$

$X_{i+1}: \tilde{5} \rightarrow a\tilde{5}$ the same eigenvalue

and $X_j \tilde{5} = \delta_j \tilde{5}$ for any $j \neq i, i+1$

$\Rightarrow \tilde{5}$ is a common eigenvector of $X_i, X_{i+1}, X_j$.

$\Rightarrow \tilde{5} \in$ G T basis

(of the same irrep $V_i$)

The vector of eigenvalues of $\tilde{5}$ is $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n)$

$\uparrow$ transpose

$\lambda_i$ and $\lambda_{i+1}$ in vector $\underline{\lambda}$.

If $b = a \pm 1$ then $5, \tilde{5}$ are linearly dependent.

$\Rightarrow \tilde{5}$ is a linearly dependent on $5 \oplus s_i(\tilde{5})$.

But we assume that $5 \oplus s_i(\tilde{5})$ are independent.

So we cannot have $b = a \pm 1$ in this case.
We obtain

**Theorem.** Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \text{Spec}(\mathbb{F}) \) correspond to vector \( \mathbf{s}_i \) in GT-basis.

- \( \alpha_1 = 0 \)
- \( \alpha_i \neq \alpha_{i+1} \quad \forall \ i \)
- If \( \alpha_{i+1} = \alpha_i \pm 1 \)
  \( \text{then} \quad \mathbf{s}_i (\mathbf{s}_i) = \mp \mathbf{s}_i \)
- If \( \alpha_{i+1} \neq \alpha_i \pm 1 \)
  \( \text{then} \quad \alpha = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_{i+2}, \ldots) \in \text{Spec}(\mathbb{F}) \)

\( \alpha \sim \alpha \)

\( \alpha \) corresponds to basis vector \( \mathbf{s}_i \)

\( s_i \) preserves the 2-dim subspace \( \langle \mathbf{s}_i, \mathbf{s}_i \rangle \)

We cannot have \( \alpha = (\alpha_1, \ldots, \alpha_n) = (\ldots, a, a\pm 1, a, \ldots) \)

**Proof.** We already proved everything except the last claim.

Last claim. Suppose

\( \alpha = (\ldots, a, a+1, a, \ldots) \)

\( s_i s_i s_i (\mathbf{s}_i) = - \mathbf{s}_i \)

\( s_i s_i s_i s_i (\mathbf{s}_i) = \mathbf{s}_i \)

**Contradiction.** \( \square \)
Claim: The above conditions uniquely describe the set $\text{Spec}(n)$ up to equiv. rel. $\sim$.

Let's be more specific.

Define an allowed transposition as $(\alpha_1, \ldots, \alpha_n) \leftrightarrow (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n)$ if $\alpha_{i+1} \neq \alpha_i \pm 1$.

Define $\text{Cont}(n) \subset \mathbb{Z}^n$ the set of vectors $(\alpha_1, \ldots, \alpha_n)$ such that

- $\alpha_1 = 0$
- If $\alpha \leftrightarrow \alpha'$ is an allowed transposition then $\alpha' \in \text{Cont}(n)$
- If $\alpha_i = \alpha_j = a, i \neq j$, then $a-1, a+1 \in \{\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_j\}$

Let $\sim$ be an equiv. rel. on $\text{Cont}(n)$ generated by allowed transpositions.
We proved that
\[ \text{Spec}(n) \subseteq \text{Cont}(n) \]
\[ \alpha \sim \alpha' \implies \alpha \approx \alpha' \]

**Theorem** The set \( \text{Cont}(n) \)
is in bijection with \( \text{stand. Young tableaux of shape } \lambda \vdash n \)
\[ \alpha \approx \alpha' \iff \text{corr esp. SYT}'s have the same shape} \]
So \# \text{\ - equiv. classes} \]
\[ = p(n) \ (\# \text{ Young tableaux}) \]

On the other hand.
\[ \# \text{\ - equiv. classes in Spec}(n) \]
\[ = \# \text{ image of } S_n = p(n) \]

So we get
**Theorem** \( \text{Spec}(n)/\sim \]
\[ = \text{Cont}(n)/\sim \]