last time:  \( S_n (\lambda_1, \ldots, \lambda_n) = \) # of STY's with top row \((\lambda_1, \ldots, \lambda_n)\) \\

= # (integer) Gelfand-Tsetlin patterns with top row \((\lambda_1, \ldots, \lambda_n)\)

Stanley's hook-content formula

\[
\prod_{(i,j) \in \lambda} \frac{n + c_{ij}}{\nu_i + \lambda_j - c_{ij}} \quad \text{hook length}
\]

\[ c_{ij} = j - i \quad \text{content of box } (i,j) \]

Weyl's dimension formula

\[
\prod_{1 \leq i < j \leq n} \lambda_i - \lambda_j + j - i
\]

Example: \(\lambda = (3, 2, 0)\)

\[
S_{320} (3, 2, 0) = \begin{pmatrix}
9 & 3 & 1 \\
2 & 1
\end{pmatrix}
\]

hook lengths

\[
\begin{pmatrix}
2+0+3+2 \end{pmatrix} \cdot \begin{pmatrix}
3+0+2 \end{pmatrix} \cdot \begin{pmatrix}
2+0+1 \end{pmatrix}
\]

contents

\[
= \begin{pmatrix}
3-2+1 \end{pmatrix} \cdot \begin{pmatrix}
3-0+2 \end{pmatrix} \cdot \begin{pmatrix}
2-0+1 \end{pmatrix}
\]

= 15.

15 STY's:

\[
\begin{pmatrix}
1 & 2 & 2 & 3 & 4 & 5 & \ldots & 2 \end{pmatrix}
\]

Which formula is "better"?

- Stanley's: product over boxes at \(\lambda\)
- Weyl's: product of \((\gamma_{ij})\) terms

If \(\lambda\) is fixed and \(n\) is large, then hook-content formula is more efficient.

If \(n\) is fixed & \(\lambda = (\lambda_1, \ldots, \lambda_n)\) an arbitrary partition, then Weyl's dim formula is more efficient.
Example \( \lambda = (n-1, n-2, \ldots, 0) \) = \[
\begin{array}{c}
\vdots \\
. \\
. \\
. \\
. \\
. \\
. \\
\end{array}
\]

"stair case shape"

# SSYT's of shape \((n-1, n-2, \ldots, 0)\) with entries \(\in \{1, \ldots, n\}\)

Weyl's formula \[
\prod_{1 \leq i < j \leq n} \frac{j-i + j-i}{j-i} = \frac{n!}{(n-2)!} \]

Ex. \( n=2 \): \[\begin{array}{c}
1 \\
2 \\
\end{array}\] \[\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\end{array}
\end{array}\]

\( n=3 \): \[\begin{array}{cccc}
1 & 1 & 1 & 2 \\
1 & 2 & 2 & 3 \\
2 & 3 & 3 & 3 \\
\end{array}\]

\[8 = 2^3 \]

Q: Is there a bijective proof?
Goldman-Totkin polytope:

\[ \lambda = (\lambda_1, \ldots, \lambda_n) \]

\[ \text{GT}(\lambda) \subset \mathbb{R}^n \] - the polytope of real-valued GT-patterns with top row \( \lambda \).

\[ \#(\text{GT}(\lambda) \cap \mathbb{Z}^n) = S_{\lambda}(\frac{1}{\rho}) \]

Ehrhart polynomial of a polytope \( P \subset \mathbb{R}^n \)

\[ L_P(t) = \# \left( tP \cap \mathbb{Z}^n \right), \quad t \in \mathbb{Z}_+ \]

Theorem (Ehrhart) If \( P \) is a lattice polytope (i.e., all vertices \( \in \mathbb{Z}^n \)), then \( L_P(t) \) is a polynomial function in \( t \).

Remark. In general, \( L_P(t) \) may not be a polynomial in \( t \) when \( P \) has non-integer vertices.

If \( P \) is a rational polytope, then \( L_P(t) \) is a quasi-polynomial (i.e., there might be several cases & several different polynomials).

But there are examples of polytopes which are not lattice polytopes, for which \( L_P(t) \) is a polynomial in \( t \). (For example, Littlewood-Richardson polytopes aka hive / honeycomb polytopes that we'll discuss later.)
Definitions of $\text{GT}$-polytopes:

\[ t \cdot \text{GT}(\lambda) = \text{GT}(t \cdot \lambda), \]

where $t \cdot \lambda := (t\lambda_1, t\lambda_2, \ldots, t\lambda_n)$.

Weyl's character formula $\Rightarrow$

**Corollary.** The Ehrhart polynomial of the Belfond-Tsetlin polytope is

\[
\text{Ehr}(t) = \prod_{1 \leq i < j \leq n} \frac{t(\lambda_i - \lambda_j) + j - i}{j - i}.
\]

In particular, $\text{Ehr}(t)$ is a polynomial in $t$ with positive coefficients.

Typically, Ehrhart polynomials don't have positive coefficients. But for some special classes of polytopes they do have positive coeffs., e.g., Ehrhart positivity holds for permutohedra and (conjecturally) for generalized permutohedra.
Example. The vertex set of the quadric

polytope \( A = (2, 2, 2, 2, 2) \)

is

\[
\text{Vol}(G) = \frac{1}{2} |G|
\]

Volume = \[
\text{Volume of an n-dimensional unit cube}
\]

For \( a = 3 \),

\[
\text{Volume of a 3-dimensional unit cube is 1.}
\]

Ponder: Construct a piecewise-linear volume preserving bijection

\[
G_1(x, y, z, w) = (0, 0, 0, 1)
\]

This is a little polytope.

Theorem. For \( A = (a_0, a_1, \ldots, a_m) \),

\( G_1(x) \) is a little polytope.

Proof. \( G_1(x) \) is given by

\[
\begin{cases}
\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m \\
\leq a_0
\end{cases}
\]

In order to describe any vertex, we need to replace some of these inequalities by equalities. We thus get

\[
\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m = a_0
\]

If we replace one more \( x_i \) by \( \leq 1 \), there will be no solutions.

So each vertex of \( G_1(x) \) has the form as in the example.

\[
\begin{cases}
\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m = a_0 \\
\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m \leq a_0
\end{cases}
\]

Each entry equals \( 1 \) or \( 0 \) of the two extreme elements.

In particular, each entry equals \( 1 \) or \( 0 \) of the two extreme elements.

Thus, all vertices of \( G_1(x) \) are integer.

Example: 4 vertices of \( G_1(2,1) \):

\[
\begin{align*}
0 & \quad 0 & \quad 0 & \quad 1 \\
1 & \quad 0 & \quad 0 & \quad 1 \\
0 & \quad 1 & \quad 0 & \quad 1 \\
0 & \quad 0 & \quad 1 & \quad 1
\end{align*}
\]
A more refined Belfand-Tsetlin polytope. Fix $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$

$$\text{GT}(\lambda, \mu) = \text{the polytope of real-valued GT-patterns with top row } \lambda \text{ and weight } \mu,$$

i.e. with row sums

$$M_1, M_1 + M_2, M_1 + M_2 + M_3, \ldots$$

(from the bottom)

**Example** $n = 3$

$$\text{GT}(\lambda, \mu) = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l}
\lambda_3 x = \lambda_2, \\
\lambda_2 z = \lambda_3, \\
x \geq z \geq y \\
z = \mu_1, \\
x + y = \mu_1 + \mu_2 \end{array} \right\}$$

Assume $\lambda, \mu$ are integer vectors.

The number of integer lattice points in $\text{GT}(\lambda, \mu)$ is $\# \text{GT}(\lambda, \mu) \cap \mathbb{Z}^3$

$$= K_{\lambda \mu} \quad (\text{Kostka number})$$

# SSYT’s of shape $\lambda$ & weight $\mu$. 

$$= \# \text{SSYT’s of shape } \lambda \text{ & weight } \mu.$$
Some questions to think about:

- Is \( G_T(x, y) \) always a lattice polytope?

- Is its Ehrhart "polynomial"

\[
G_T(x, y) (t) = K_{t \lambda, t \mu}
\]

a polynomial in \( t \)?

- Does it have positive coefficients?

Examples: \( n = 2 \), if \( K_{\lambda \mu} \neq 0 \).

(i.e. if \( \lambda_1, \mu_1 > \lambda_2, \mu_2 \))

then \( K_{t \lambda, t \mu} = 1 \)  \( \forall t \in \mathbb{Z}_{\geq 0} \).

\( n = 3 \)

\[ \begin{align*}
\lambda_1 & \quad \lambda_2 \quad \lambda_3 \\
X & \quad Y \\
\end{align*} \]

\[ \begin{align*}
x + y & = \mu_1 + \mu_2 \\
x & = \lambda_1 \\
y & = \lambda_2 \\
\end{align*} \]

If \( K_{\lambda \mu} = \ell + 1 \) then

\[ \begin{align*}
K_{t \lambda, t \mu} & = (t + 1)^{\ell} \\
\end{align*} \]

This is a polynomial

in \( t \) with positive coefficients.

We can answer some of the above questions using Kostant's partition function...
Kostant's partition function $P(\beta)$

(ot type A)

$\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ any vector,

s.t. $\beta_1 + \ldots + \beta_n = 0$ (not a partition)

$P(\beta) := \# \text{ ways to express } \beta \text{ as a nonnegative integer linear combination of vectors } e_i - e_j,$

$1 \leq i < j \leq n$

(called positive roots)

(Here $e_1, \ldots, e_n$ are the coord. vectors in $\mathbb{R}^n$)

Ex. $n = 3$

$\beta = (2, 0, -2)$

$\beta = 2(e_1 - e_3) = (e_1 - e_3) + (e_1 - e_3) + (e_2 - e_3)$

$= 2(e_1 - e_3) + 2(e_2 - e_3)$

So $P((2, 0, -2)) = 3.$
Kostant multiplicity formula

(for type A)

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \mu = (\mu_1, \ldots, \mu_n) \),

\[
K_{\lambda \mu} = \sum_{w \in S_n} (-1)^{\text{wt}(w)} P(w(\lambda+\rho) - (\mu+\rho)),
\]

where \( \rho = (n-1, n-2, \ldots, 1, 0) \).

Kostka numbers are known as weight multiplicities in Lie theory.

The same thing that we earlier denoted \( \delta \).

Example. \( n = 3 \)

In lecture 4, we calculated

\[
S_{(4,2,0)}(x_1, x_2, x_3) = \ldots + 3 x_1^2 x_2^2 x_3^2 + \ldots
\]

\[
K_{(4,2,0), (2,2,2)} = 3
\]

Kostant's formula: \( K_{(4,2,0), (2,2,2)} = \)

\[
= p(2,0,2) - p(-1,3,-2) - p(-1,3,2)
\]

\[
= 3, 0, 0 + \ldots
\]

\[
= 3.
\]
Proof. Classical def. of $S_\lambda(x_1, x_2)$

$$S_\lambda := \frac{a_{\lambda p}}{a_p} = \sum_{\iota \in \mathbb{S}_n} (-1)^{\iota(w)} w(\lambda + p) \times \frac{X_p}{\prod_{i \leq j \in \iota} (1 - \frac{x_i}{x_j})}$$

$$= \sum_{\iota \in \mathbb{S}_n} (-1)^{\iota(w)} w(\lambda + p) \times p \prod_{i \leq j \in \iota} \frac{1}{1 - \frac{x_i}{x_j}}$$

We have \(\prod_{i \leq j \in \iota} \frac{1}{1 - \frac{x_i}{x_j}} = \prod_{i < j} \frac{1}{1 - x_i x_j} = \prod_{i < j} 1 - (x_i - x_j) = \prod_{i < j} (\sum_{k \geq 0} x_i^k (x_j - x_i))\)

$$= \sum_{\rho \in \mathbb{Z}^n} p(\rho) x^{\rho}$$

This is basically the def. of Kostant's partition function

$$S_\lambda = \sum_{\iota \in \mathbb{S}_n} (-1)^{\iota(w)} w(\lambda + p) - p \times \rho \in \mathbb{Z}^n$$

$$= \sum_{\iota \in \mathbb{S}_n} (-1)^{\iota(w)} p(w(\lambda + p) - \rho) \times X^\rho$$

$$M \in \mathbb{Z}^n$$

So \(k_{\lambda, \mu} := \) the coeff. of \(x^\mu \) in \(S_\lambda = \)

$$= \sum_{\iota \in \mathbb{S}_n} (-1)^{\iota(w)} p(w(\lambda + p) - \rho) \times \rho \in \mathbb{Z}^n$$

\(\blacksquare\)
\( P(\beta) \) is \# of integer lattice points of \text{flow polytope} 

\[
\text{Flow}(\beta) := \left\{ (x_{ij})_{1 \leq i \leq n, j \leq n} \in \mathbb{R}_{+}^{n^2} \mid \begin{array}{l}
x_{ij} \geq 0 \\
\sum x_{ij} (e_i - e_j) = \beta
\end{array} \right\}
\]

We can think of \((x_{ij})\) as flows along edges \((i,j)\) of \(K_n\) (directed \(i \rightarrow j\) for \(i \leq j\)) 
s.t. A vertex \(k \in [n] \)

\[
\sum_{i \rightarrow k} x_{ik} + \beta_k = \sum_{j \leftarrow k} x_{kj}
\]

\((\beta_1, \ldots, \beta_n)\) is \text{excess flow vector}.

Example: \(\beta = (2, 0, 0, -2)\)

\[
2 \rightarrow 1 \quad 2 \quad 1 \quad 2 \quad 1 \quad 2 
\]

\(2\) \text{ flow on } K_4

Lemma: \text{Flow}(\beta) \text{ is an lattice polytope in } \mathbb{R}^{n^2} \) (all vertices are integer vectors)

Exercise: Prove this by describing the vertices of \text{Flow}(\beta).
Example.

Let's describe vertices of Flow $(\beta)$ for $\beta = (1, 0, 0, \ldots, 0, -1)$

Flow $(1, 0, \ldots, 0, -1)$ has $2^{n-2}$ vertices that correspond to all directed paths in $K_n$ from 1 to n (with edges directed as $i \to j$ for $i < j$).

![Diagram of directed graph]

A flow in $K_7$ with excess in-flow 1 for the 1st vertex and excess out-flow 1 for the last vertex.

It is not hard to see that any other flow in $K_n$ for $\beta = (1, 0, \ldots, 0, -1)$ is a non-negative linear combination of such "path flows".
Theorem. Fix $\beta, \lambda, \mu \in \mathbb{Z}^n$.

1. Kostant's partition function $p(t \cdot \beta)$ is a polynomial in $t$.

2. Kostka number (or weight multiplicities)

\[ K_{\lambda, \mu} \text{ is a polynomial in } t. \]

In both cases, we assume that $t$ is a positive integer.

Proof. (1) Follows from Ehrhart Thm.

(2) Follows from (1) & Kostant's formula. \[ \square \]

Remark. This argument proves polynomiality, but does not prove positivity.

Remark. Kostant's partition function $p(\beta)$ and Kostka numbers $K_{\lambda, \mu}$ considered as multivariate functions of $\beta_1, \ldots, \beta_n$ and $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ are piecewise polynomial functions.
Conjecture $p(\beta)$ is a polynomial in \(\beta\) with positive coefficients.

Theorem. This conjecture holds for \((\beta_1, ..., \beta_n)\) s.t.
\[
\beta_1, \beta_2, ..., \beta_{n-1} \geq 0.
\]
(In this case, there is an explicit polynomial formula for $p(\beta)$
Lidskii, P.,-Stanley, Baldoni-Vergne)

A more general conjecture

**Conjecture.** $K_{t+n, t'}$ is a polynomial in $t$ with positive coefficients.

**Why "more general"?**

**Lemma.** $p(\beta) = K_{p+n, np}$ for sufficiently large $N$.

**Proof.** Kostant's formula for $K_{p+n, np}$ has only one term $P(\beta)$ if $N$ is large. Indeed,

\[
K_{p+n, np} = \sum_{w \in S_n} (-1)^{\ell(w)} P(w(\beta+np)+p-Np)
\]

for $w = \text{id}$, we get $P(\beta)$

for all other $w's$ we get

\[
P(\text{some vector that cannot be expressed as a non-negative combination of positive roots } e_i - e_j) = 0
\]

\[
\geq w(Np) - np
\]
even more general conjecture...

Later in this course we’ll talk about the Littlewood-Richardson coefficients $c_{\lambda \mu}^{\nu}$:

$(\lambda, \mu, \nu$ partitions) defined by

$$S_{\lambda} \cdot S_{\mu} = \sum c_{\lambda \mu}^{\nu} S_{\nu}$$

The LR-coeffs $c_{\lambda \mu}^{\nu}$ include the Kostka numbers $K_{\lambda \mu}$ as special cases:

Kostka's partition functions can also be expressed as linear combinations of Kostka's partition functions $K_{\lambda \mu}$

Conjecture (King-Tolly-Toumazet)

For any partitions $\lambda, \mu, \nu$

$$c_{\lambda \mu}^{\nu + \lambda} + c \in \mathbb{Z}_{>0}$$

is a polynomial $\hat{f}(t)$ with positive integer coeffs., or $\hat{f}(t) \equiv 0$. 
The previous conjecture can be viewed as a strong form of the Saturation Conjecture.

Saturation Conjecture (proved by Knutson-Tao based on works of Horn, Klyachko, Berenstein-Zelevinsky)

If $C_{t, \lambda, t' \mu} \neq 0$ for some $t$, then $C_{\lambda, \mu} \neq 0$.

Clearly, if $C_{t, \lambda, t' \mu}$ is a polynomial $f(t)$ in $t$ with positive coefficients, then Saturation holds: if one value of $f(t)$ (for $t > 0$) is non-zero, then all values of $f(t)$ (for $t > 0$) are non-zero.

We'll talk more about this stuff later in this course...