18.211 LECTURE 5 9/15

Last Time: Operators acting on $\mathbb{C}[x_1, \dots, x_n]$

- div. diff operators $\partial_i : f \longmapsto \frac{1}{x_i - x_{i+1}}(1 - s_i) f$
- Demazure operators $D_i : f \longmapsto \frac{f - \frac{x_{i+1}}{x_i} s_i(f)}{1 - \frac{x_{i+1}}{x_i}}$

Let $X_i : f \longmapsto x_i f$

Then $\boxed{D_i = \partial_i \circ X_i}$ i.e. $D_i(f) = \partial_i(x_i f)$

For $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ reduced decomp. of w

$\partial_w := \partial_{i_1} \cdots \partial_{i_\ell}$ and $D_w := D_{i_1} \cdots D_{i_\ell}$

2 formulas for Schur polys

$S_\lambda(x_1, \dots, x_n) = \partial_{w_0}(x^{\lambda + \delta}) \overset{?}{=} D_{w_0}(x^\lambda)$ $\qquad \delta = (n-1, n-2, \dots, 0)$

Thrm: $\partial_{w_0} \circ X_1^{n-1} X_2^{n-2} \cdots X_n^0 = D_{w_0}$

E.x. $n = 3$ $\qquad w_0 = s_1 s_2 s_1 \in S_n$

$\partial_1 \partial_2 \partial_1 X_1 X_1 X_2 \overset{?}{=} \overbrace{\partial_1 X_1}^{D_1} \overbrace{\partial_2 X_2}^{D_2} \overbrace{\partial_1 X_1}^{D_1}$ $\quad (= \partial_2 X_2 \partial_1 X_1 \partial_2 X_2)$

Problem: ∂_i's don't always commute with D_j's

Lemma:
- $D_i X_j = X_j D_i$ if $j \notin \{i, i+1\}$
- $\partial_i (X_i X_{i+1}) = (X_i X_{i+1}) \partial_i$

Why:
- D_i only affects x_i and x_{i+1}, doesn't affect any others so for other j we can commute
- D_i is symmetric wrt x_i and x_{i+1}

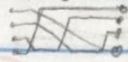
E.x. $\partial_1 X_1 \partial_2 X_2 \partial_1 X_1$

$= \partial_1 \partial_2 (X_1 X_2) \partial_1 X_1$

$= \partial_1 \partial_2 \partial_1 X_1 X_2 X_1 = \partial_1 \partial_2 \partial_1 X_1^2 X_2$

Proof (for general n):

Write $w_0 = (s_1 s_2 \dots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2)(s_1)$ reduced decomp

wiring diagram 

$D_{w_0} = (\partial_1 X_1 \partial_2 X_2 \cdots \partial_{n-1} X_{n-1})$ $\qquad = (\partial_1 \partial_2 \cdots \partial_{n-1} X_1 X_2 \cdots X_{n-1})$

$(\partial_1 X_1 \partial_2 X_2 \cdots \partial_{n-2} X_{n-2})$ $\qquad (\partial_1 \partial_2 \cdots \partial_{n-2} X_1 X_2 \cdots X_{n-2})$

$\vdots$ $\qquad\qquad \vdots$

$(\partial_1 X_1 \partial_2 X_2)$ $\qquad (\partial_1 \partial_2 X_1 X_2)$

$(\partial_1 X_1)$ $\qquad (\partial_1 X_1)$

Can move all X's in each line to the right

Now all combined X terms symetric w.r.t operators on the right, so we can commute them to get exactly what we want.

Ex. $n=3$

$w_0: D_{w_0} = \partial_{w_0} x_1^2 x_2$ $\qquad s_1: \partial_1 x_1,\ s_2: \partial_2 x_2,\ 1=1$

$s_1 s_2: \ \partial_1 x_1 \partial_2 x_2 = \partial_1 \partial_2 x_1 x_2$

BUT $s_2 s_1: \ \partial_2 x_2 \partial_1 x_1 \neq \partial_2 \partial_1 x_2 x_1$ $\qquad s_2 s_1 = 312$

Claim: Can write the ∂_i first then x_i iff w is 312 avoiding

Schubert Poly $\mathfrak{S}_u = \partial_{u^{-1}w_0}(x^\delta)$

Demazure char $ch_{\lambda,w} = D_w(x^\lambda)$

Question: When is $\mathfrak{S}_u = ch_{\lambda,w}$? $\qquad \lambda = (\lambda_1 \geq \cdots \geq \lambda_n),\ w \in S_n$

Thrm 1 If w is a 312-avoiding perm in S_n, then $ch_{\lambda,w} = \mathfrak{S}_u$ for some perm $u \in S_m$ (not necess. $= n$)

Thrm 2 If u is a 2143-avoiding perm in S_m, then $\mathfrak{S}_u = ch_{\lambda,w}$ for some λ, w
vexillory perms

Combinatorial Def of S_λ:

$$S_\lambda(x_1,\dots,x_n) = \sum_{\substack{T \text{ SSYT} \\ \text{of shape } \lambda}} x^{\text{weigh}(T)}$$

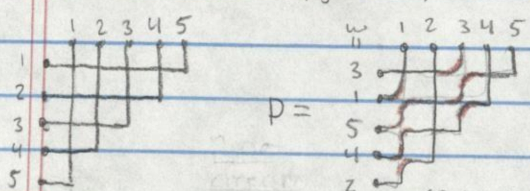
Ex.
```
11111 333
2223
4
```
weight $= (\#1\text{'s}, \#2\text{'s}, \dots, \#n\text{'s})$

Combinatorial formula for $\mathfrak{S}_w$

RC-graphs aka pipe dreams

[Formin-Stanley, Billey-Jokushi-Stanley]

Def: Pipe Dream



$\beta = (3,1,1,0,0)$

No double crossings

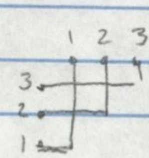
Any two wires intersect at most once

Def: weight$(P) = (\beta_1, \beta_2, \dots, \beta_n)$

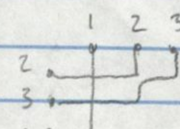
$\beta_i = \#$ crossings in i^{th} row

Thrm: $\mathfrak{S}_w = \sum_{P = \text{pipe dream}} x^{\text{weight}(P)}$

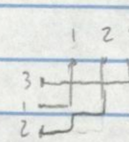
Ex. $n=3$ $\quad \mathfrak{S}_{w_0} = x_1^2 x_2$



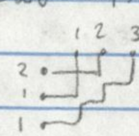
$\mathfrak{S}_{231} = x_1 x_2$



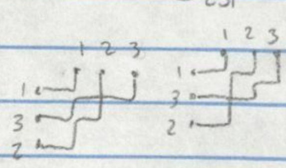
$\mathfrak{S}_{312} = x_1^2$



$\mathfrak{S}_{213} = x_1$



$\mathfrak{S}_{132}$



Two options $\Rightarrow \mathfrak{S}_{132} = x_1 + x_2$

Schubert Polynomials $\mathfrak{S}_w$, $w \in S_n$

- Defined div. diff. operators ∂_i

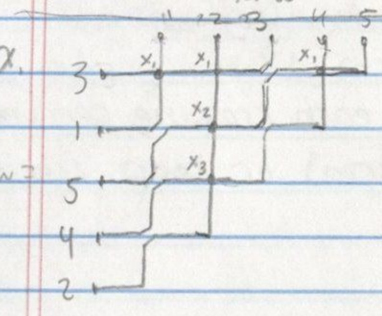
(1) $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$

(2) $\mathfrak{S}_w = \partial_i(\mathfrak{S}_{ws_i})$ if $\ell(ws_i) = \ell(w) + 1$

- A combinatorial rule for $\mathfrak{S}_w$ via pipe dreams

Thrm: [Billey-Jokush-Stanley] cf [Fomin-Stanley]

$$\mathfrak{S}_w = \sum_{P \text{ pipe dream for } w} x^{\text{weight}(P)}$$

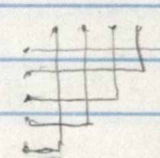
Ex. 

it should be reduces (any 2 wires intersect at most once)

$w = 31542$ $x^{\text{weight}(P)} = x_1^3 x_2 x_3$

Nil Coxete algebra N_n (over $\mathbb{C}$)

- generators $u_1, u_2, \ldots, u_{n-1}$
- relations
  - (1)' $u_i^2 = 0$
  - (2) $u_i u_j = u_j u_i$, $|i-j| \geq 2$ ← Note: Same relations as ∂_i's
  - (3) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

Today: Want to prove Thrm.
Show formula satisfies 2 relation
(1) is "obvious". Just draw pipedream diagram for w_0 
(There is exactly 1 way to do This)

- linear basis of N_n u_w, $w \in S_n$

$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ (reduced decomp)

$u_w := u_{i_1} u_{i_2} \cdots u_{i_\ell}$

NOTE! If $s_{i_1} \cdots s_{i_\ell}$ is not reduced then $u_{i_1} \cdots u_{i_\ell} = 0$.

$v, w \in S_n$

$$u_v \cdot u_w = \begin{cases} u_{vw} & \text{if } \ell(u \cdot w) = \ell(u) + \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

Commutative variables $x_1, x_2, \ldots, x_{n-1}$ that also commute with u_i's

$h_i(x) := 1 + x u_i$

$A_i(x) := h_{n-1}(x) h_{n-2}(x) \cdots h_i(x)$ $\mathfrak{S} := A_1(x_1) A_2(x_2) \cdots A_n(x_n)$

<u>Thrm</u>: [F·S]

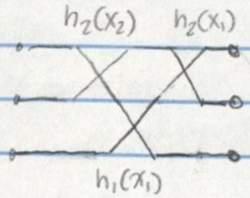
$$\mathfrak{S} = \sum_{w\in S^n} \mathfrak{S}_w(x_1, x_2, \ldots, x_n)\, u_w$$ ← The schubert poly

<u>Ex.</u> $n=3$ $\quad \mathfrak{S} = A_1(x_1)A_2(x_2)$

$= \underline{h_2(x_1)h_1(x_1)}\,\underline{h_2(x_2)}$

$= (1+x_1u_2)(1+x_1u_1)(1+x_2u_2)$

$= \underbrace{1}_{\mathfrak{S}_{id}} + \underbrace{x_1u_1}_{\mathfrak{S}_{s_1}} + \underbrace{(x_1+x_2)u_2}_{\mathfrak{S}_{s_2}} + \underbrace{x_1x_2u_1u_2}_{\mathfrak{S}_{s_1s_2}} + \underbrace{x_1^2u_2u_1}_{\mathfrak{S}_{s_2s_1}} + \underbrace{x_1^2x_2u_2u_1u_2}_{\mathfrak{S}_{s_2s_1s_2}}$



For w_0. And the for each crossing can undo it (corresponds to the 1 term) or keep (corresponds to x_iu_j term)

Let $\mathfrak{S} = \sum_{w\in S_n} \tilde{\mathfrak{S}}_w(x_1,\ldots,x_{n-1})\,u_w$

WTS $\tilde{\mathfrak{S}}_w = \mathfrak{S}_w$

Need to check (1) $\tilde{\mathfrak{S}}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$

(2) $\tilde{\mathfrak{S}}_w = \partial_i(\tilde{\mathfrak{S}}_{ws_i})$ if $\ell(ws_i) = \ell(w)+1$

We need to prove the identity

$\boxed{(*)\ \partial_i(\mathfrak{S}) = \mathfrak{S}\cdot u_i}$

$$\partial_i(\mathfrak{S}) = \sum_{w\in S_n} \partial_i(\tilde{\mathfrak{S}}_w)\,u_w$$

$$\mathfrak{S}u_i = \sum_{\substack{w\in S_n\\ \ell(ws_i)=\ell(w)+1}} \tilde{\mathfrak{S}}_w\, u_{ws_i}$$

<u>Proof of (*)</u> $\partial_i(A_1(x_1)A_2(x_2)\cdots A_i(x_i)A_{i+1}(x_{i+1})\cdots A_{n-1}(x_n))$

NTS $= A_1(x_1)A_2(x_2)\cdots A_i(x_i)A_{i+1}(x_{i+1})\cdots A_{n-1}(x_n)\,u_i$

$\boxed{\text{ETS } \partial_i(A_i(x_i)A_{i+1}(x_{i+1})) = A_i(x_i)A_{i+1}(x_{i+1})\,u_i \quad (**)}$

<u>Lemma 1</u>: (1) $h_i(x)h_i(y) = h_i(x+y)$ $\qquad h_i(0)=1$

(2) $h_i(x)h_j(y) = h_j(y)h_i(x)$ if $|i-j|\geq 2$

(3) $h_i(x)h_{i+1}(x+y)h_i(y) = h_{i+1}(y)h_i(x+y)h_{i+1}(x)$

← <u>Yang-Baxeter relations</u>

<u>Lemma 2</u>: $A_i(x)$ & $A_i(y)$ commute with each other

$A_i(x)A_i(y) = A_i(y)A_i(x)$ (expression symetric in x & y)

<u>Lemma 3</u>: $A_i(x)A_{i+1}(y)\,u_i = A_i(y)A_{i+1}(x)\,u_i$ (" ")

<u>Lemma 4</u>: $A_i(x)A_{i+1}(y) - A_i(y)A_{i+1}(x) = (x-y)A_i(x)\cdot A_{i+1}(y)\cdot u_i$

Lemma 4 $\Rightarrow$ (**)

<u>E.x.</u> Of Lemma 2 $\quad n=3,\ i=2$

$$h_2(x)\,h_2(y) = h_2(x+y) = h_2(y) + h_2(x)$$

$$\underbrace{h_2(x)h_1(x)}_{A_1(x)}\ \underbrace{h_2(y)h_1(y)}_{A_1(y)}$$

$$= h_2(x)h_1(x)h_2(y)\,h_1(y-x)\,h_1(x-y)h_1(y)$$

$$= h_2(x)h_2(y-x)h_1(y)h_2(x)\ h_1(x)$$

$$= h_2(y)h_1(y)\,h_2(x)h_1(x)$$

Schub Polys $\to$ div. diff operators
$\to$ pipe dreams

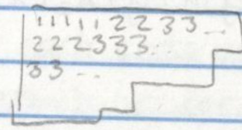
$\lambda = (\lambda_1, \dots, \lambda_k) \subset k \times (n-k)$ rectangle
partition
$\rightsquigarrow w(\lambda) \in S_n$ Grassmanian perm. $w(\lambda) = \lambda_k + 1, \lambda_{k-1} + 2, \dots, \lambda_1 + k$
(all other indices in incr. order)

Then $s_\lambda(x_1, \dots, x_k) = \mathfrak{S}_{w(\lambda)}$

Comb. Formula for $s_\lambda(x_1, \dots, x_k)$

Thrm: $s_\lambda(x_1, \dots, x_k) = \sum_{T \in SSYT(\lambda)} x^{wt(T)} \longleftarrow x_1^{\#1s} x_2^{\#2s} \cdots$

Recall: $SSYT(\lambda)$ = semi-standard Young tableaux of shape λ



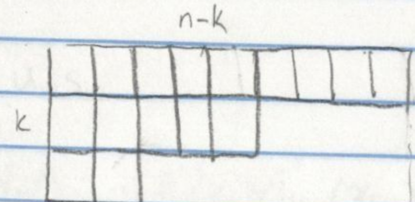
Pipe dream formula for $\mathfrak{S}_w$ $\overset{?}{\rightsquigarrow}$ SSYT formula for s_λ

(Want to see SSYT formula as special case of pipe dreams.

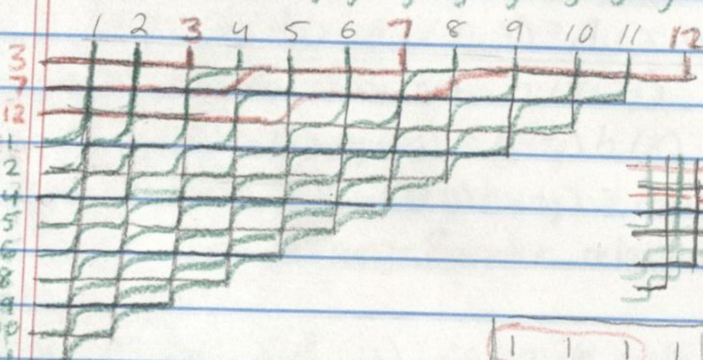
Claim: Pipe dreams P for $w(\lambda)$ are in bij. with $SSYT(\lambda)$

Rule: Order the wires of P from top to bottom by their left end
For $i = 1, \dots, k$,
\# crossings of i^{th} wire of P in j^{th} row $=$ \# entries $k+1-j$ in $(k+1-i)^{th}$ row of T

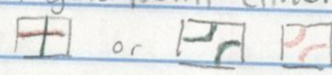
Ex.: $n = 12$, $k = 3$
$\lambda = (9, 5, 2)$

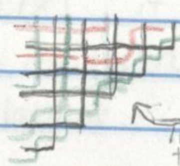


$w(\lambda) = 3, 7, 12, 1, 2, 4, 5, 6, 8, 9, 10, 11, 12$



each grid point either





(Should be drawn inside boxes, oops) like this

1	1	1	1	2	2	3	3	3
2	2	2	3	3				
3	3							

Always get

have choices

(Calculate rows from bottom up)

<u>Thm</u>: $S_\lambda(x_k, x_{k-1}, \dots, x_1) = \sum_{T \in SSYT(\lambda)} x^{wt(T)}$

<u>Lemma</u>: $S_\lambda = (x_1, \dots, x_k) = \mathfrak{S}_{w(\lambda)}$ is symetric in $x_1, \dots, x_k$

<u>Lemma</u>: $\mathfrak{S}_w$ is symetric in x_i & x_{i+1} if $\ell(ws_i) = \ell(w)+1$

Double Schubert Polynomials

$\mathfrak{S}_w(x,y) := \mathfrak{S}_w(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$ $\quad w \in S_n$

Almost do the same as for usual Schub

(1)' $\mathfrak{S}_{w_0}(x,y) = \prod_{\substack{(i,j) \\ i,j \geq 1 \\ i+j \leq n}} (x_i - y_j)$

(2) $\mathfrak{S}_w(x,y) = \partial_i^* \mathfrak{S}_{ws_i}(x,y)$ $\quad$ if $\ell(ws_i) = \ell(w)+1$

$\partial_i^* : f \longmapsto \frac{1}{x_i - x_{i+1}}(f(x_1, \dots, x_i, x_{i+1}, \dots) - f(x_1, \dots, x_{i+1}, x_i, \dots))$

<u>Prop</u>: Positivity: $\mathfrak{S}_w(x_1, \dots, x_{n-1}, -y_1, -y_2, \dots, -y_{n-1})$ has positive int. coeffs.

Pipe dream formula for $\mathfrak{S}_w(x,y)$

<u>Thrm</u>: $\mathfrak{S}_w(x,y) = \sum_{\substack{P \text{ pipe dream} \\ \text{for } w}} \prod_{\substack{(i,j) \text{ s.t. } P \text{ has a} \\ \text{crossing in the } i^{th} \text{ row} \\ \text{\& } j^{th} \text{ column}}} (x_i - y_j)$

<u>Cor</u> (Symmetry): $\mathfrak{S}_w(x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$
$= \mathfrak{S}_w(-y_1, \dots, -y_{n-1}, -x_1, \dots, -x_{n-1})$

Algebra (over $\mathbb{C}$)

generators: $u_1, \dots, u_{n-1}$, $x_1, \dots, x_{n-1}$, $y_1, \dots, y_{n-1}$

relations: x_i, y_j commute with each other and u_k's

(1)' $u^2 = 0$

(2) $u_i u_j = u_j u_i$ $\quad |i-j| \geq 2$

(3) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

$h_i(a) = 1 + a u_i$ satisfy $\underset{\text{Yang-Baxter}}{\text{YB}}$ relations

$\mathfrak{S}^{x,y} := \prod_{\substack{(i,j) \\ i,j \geq 1 \\ i+j \leq n}} h_{i+j-1}(x_i - y_j)$ $\longleftarrow$ order (i,j)'s as

$(1,n-1)(1,n-2) \dots (1,1)$
$(2,n-2)(2,n-3) \dots (2,1)$
$(3,n-3) \dots (3,1)$
$\vdots$
$(n-1,1)$

LECTURE 9 9/27

Cauchy identity for s_λ

$$\sum_{\lambda=(\lambda_1,\dots,\lambda_n)} s_\lambda(x_1,\dots,x_n)\, s_\lambda(y_1,\dots,y_n) = \prod_{i,j\in[n]} \frac{1}{1-x_iy_j}$$

Let $x^\alpha y^\beta = x_1^{\alpha_1}\cdots x_n^{\alpha_n}\, y_1^{\beta_1}\cdots y_n^{\beta_n}$

LHS: $\underset{\text{coeff of}}{[x^\alpha y^\beta]} = \#\left\{(P,Q)\,\middle|\, P,Q \text{ SSYT of shape } \lambda \text{ s.t. } \begin{matrix}\alpha = wt(P)\\ \beta = wt(Q)\end{matrix}\right\}$

RHS: $\prod_{i,j} \frac{1}{1-x_iy_j} = \prod_{i,j} \sum_{a_{ij}\geq 0} (x_iy_j)^{a_{ij}}$

$$= \sum_{\substack{A=(a_{ij})\\ \text{noneg-int}\\ n\times n \text{ matrix}}} \prod_i x_i^{i\text{th col. sum of } A} \prod_j y_j^{j\text{th row of } A}$$

Robinson-Schensted-Knuth correspondence (RSK)

Robinson-Schensted correspondence

special case where $\alpha=\beta=(\overbrace{1,1,\dots,1}^{n})$

P,Q are SYT's of the same shape

A - permutation matrix

$$\underset{w=w_1\cdots w_n}{S_n} \xleftrightarrow{RS} \{(P,Q)\mid \text{SYT's of same shape } \lambda\vdash n\}$$

Ex. Example of RSK. $n=2$

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix} \xleftrightarrow{RSK} (P,Q)$$

$P = \begin{array}{|l|}\hline \overbrace{1\,1\cdots}^{e}\;1\,2\cdots 2 \\ \hline 2\cdots 2 \\ \hline\end{array}\quad \begin{matrix}\lambda_1\\ \lambda_2\end{matrix}$ $\qquad$ $Q = \begin{array}{|l|}\hline \overbrace{1\cdots 1}^{f}\,2\cdots 2 \\ \hline 2\cdots 2 \\ \hline\end{array}\quad \begin{matrix}\lambda_1\\ \lambda_2\end{matrix}$

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \longleftrightarrow \begin{pmatrix} \lambda_2 & \leq & e\\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$}\\ f & \leq & \lambda_1\end{pmatrix}$$

col. sum $a+c=e$ $\qquad$ row sums $a+b=f$

$b+d=\lambda_1+\lambda_2-e$ $\qquad$ $c+d=\lambda_2+\lambda_1-f$

These 4 e.g. are not lin. independent $\Rightarrow$ Not enough to solve for RSK correspondence

Actual Answer $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \longleftrightarrow \begin{pmatrix} \min(b,c) & a+c\\ a+b & a+d+\min(b,c)\end{pmatrix}$

In general can view RSK as piecewise linear tranform of matrices

Classical Construction of RSK

Ex. $A = \begin{pmatrix} 0&1&0&0\\ 2&0&0&1\\ 0&0&1&0\\ 0&0&0&0 \end{pmatrix} \rightsquigarrow$ biword w with a_{ij} entries $\binom{i}{j}$ ← a pair of entries
arranged lexicographically

$w = \binom{1}{2}\binom{2}{1}\binom{2}{1}\binom{2}{4}\binom{3}{3}$

1 in pos. (2,1) — 2 is pos (2,1) — 1 in pos (2,4)

will insert entries into P & Q using

Schnested insertion algorithm

Ex. $P = \begin{array}{|ccccccc} 1&1&1&3&3&3&8 \\ 2&2&2&4&5 \\ 3&4&6&6&7 \\ 5 \end{array}$ ← $\boxed{j=2}$

1st entry in 1st row which > j

1st entry in 2nd row which > 3

1st entry in 3rd row which > 4

$\begin{array}{|ccccccc} 1&1&1&\boxed{2}&3&3&8 \\ 2&2&2&3&5 \\ 3&4&4&6&7 \\ 5&6 \end{array}$

Back to Ex. from top

	$\binom{1}{2}$	$\binom{2}{1}$	$\binom{2}{1}$	$\binom{2}{4}$	$\binom{3}{3}$	$\binom{i}{j}$
P: $\emptyset$	insert 2 $\boxed{2}$	$\begin{array}{c} 1 \\ 2 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 2 \end{array}$	$\begin{array}{ccc} 1 & 2 & 4 \\ 2 \end{array}$	$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 \end{array}$	
Q: $\emptyset$	insert 1 $\boxed{1}$	$\begin{array}{c} 1 \\ 2 \end{array}$	$\begin{array}{cc} 1 & 1 \\ 2 \end{array}$	$\begin{array}{ccc} 1 & 1 & 2 \\ 2 \end{array}$	$\begin{array}{ccc} 1 & 2 & 2 \\ 2 & 3 \end{array}$	

- Insert j to P using Schnested insertion
- Insert i to Q in whatever box got added to P

Thrm: (Knuth) This construction $A \mapsto (P, Q)$ is bij. with needed properties.

Proof: Bijection: Can undo algorithm.
Idea: Find max entry in Q tableaux. That tells where we added in P tableaux, un-add via Schnested in reverse

Thrm: (Knuth) If $A \xmapsto{RSK} (P,Q)$, then $A^T \xmapsto{RSK} (Q,P)$

Not so clear why we should have this symmetry from this version of RSK

Thrm: (Knuth) If λ is the shape of P & Q tableaux, then

λ_1 = length of a "max <u>weakly</u> <u>increasing</u> subsequence" in w

λ_1' = length of a "max <u>strictly</u> <u>decreasing</u> subseq." in w

Def: Weakly increasing subseq.

$w = \binom{i_{a_1}}{j_{a_1}} \binom{i_{a_2}}{j_{a_2}} \cdots \binom{i_{a_\ell}}{j_{a_\ell}}$ s.t.

$i_{a_1} \le i_{a_2} \le \cdots \le i_{a_\ell}$

$j_{a_1} \le j_{a_2} \le \cdots \le j_{a_\ell}$

Strictly decreasing subseq

$i_{a_1} < \cdots < i_{a_\ell}$

$j_{a_1} > \cdots > j_{a_\ell}$

Ex. In example before, $\lambda_1 = 3$, $\lambda_1' = 2$

A different correspondence that lets us see Thrm properties better

SSYT $\longleftrightarrow$ Gelfand-Tsetlin Patterns

Ex. $n=5$

$$\begin{array}{cccccccc}
1&1&1&1&2&2&\cancel{5}&\cancel{5}\\
2&2&2&3&3&3&&\\
3&3&\cancel{4}&\cancel{4}&\cancel{5}&\cancel{5}&&\\
\cancel{5}&&&&&&&
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccccccccc}
8&&6&&6&&2&&0\\
&6&&6&&4&&0&\\
&&6&&6&&2&&\\
&&&6&&3&&&\\
&&&&4&&&&
\end{array}$$

- Write # elts in each row of T
- Cross out all of highest number
- Repeat for next row of GT-pattern