LECTURE 18 10/19

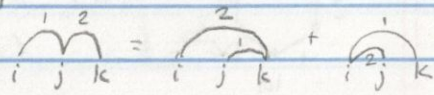
Fomin-Kirillov's algebra FK_n

<u>generators</u>: T_{ij} $\quad i,j \in [n]$, $\quad T_{ij} = -T_{ji}$, $\quad T_{ii} = 0$

<u>relations</u>: (1) $T_{ij}^2 = 0$

(2) $T_{ij} T_{k\ell} = T_{k\ell} T_{ij}$ if $\# \{i,j,k,\ell\} = 4$

(3) $T_{ij} T_{jk} = T_{jk} T_{ik} + T_{ik} T_{ij}$



(3') $T_{ij} T_{ik} + T_{jk} T_{ki} + T_{ki} T_{ij} = 0$

FK_n acts on <u>coinvariant algebra</u>

$$C_n = \mathbb{C}[x_1, \dots, x_n]/I_n = \langle \sigma_w \mid w \in S_n \rangle \quad \text{where } \sigma_w \equiv \mathfrak{S}_w \pmod{I_n}$$

$I = \langle e_1, \dots, e_n \rangle$ symmetric polys $\qquad$ <u>Schubert classes</u>

by $\underset{\text{for } i<j}{T_{ij}} : \sigma_w \longmapsto \begin{cases} \sigma_{wt_{ij}} & \text{if } \ell(wt_{ij}) = \ell(w)+1 \\ 0 & \text{otherwise} \end{cases}$

All complexity of Schubert calculus are hidden in these relations

<u>(Chevelley) Monk's Formula</u>

$\ell(x) = \ell_1 x_1 + \dots + \ell_n x_n \pmod{I_n} \in C_n$

a linear form (cohomology class of a line bundle)

$$\ell(x) \cdot \sigma_w = \sum_{\substack{(i,j) \\ 1 \le i < j \le n}} (\ell_i - \ell_j)\, T_{ij}(\sigma_w)$$

$\uparrow$ chevellay multiplicities

<u>Divided Diff</u> $\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i)$

<u>"Liebniz rule" for ∂_i</u>

$$\partial_i(\ell(x) \cdot f) = s_i(\ell(x)) \cdot \partial_i(f) + \underbrace{\partial_i(\ell(x))}_{= (\ell_i - \ell_{i+1})} \cdot f$$

<u>Lemma</u>: ∂_i are well defined on C_n

<u>Proof</u> of Monk's formula:

$\sigma_w = \partial_{w^{-1}w_0}(\sigma_{w_0})$ $\leftarrow$ top Schubert class

We know for any poly, $\ell(x) \cdot \sigma_{w_0} = 0$ (in C_n)

B/C σ_{w_0} already of top degree, anything of higher degree vanishes in C_n

$w^{-1}w_0 = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ (reduced)

$\cdot\ \ell(x) \cdot \sigma_w = \ell(x) \cdot \partial_{i_1} \cdots \partial_{i_\ell}(\sigma_{w_0})$

$= \partial_{i_1} s_{i_1} \ell(x) \cdot \partial_{i_2} \cdots \partial_{i_\ell}(\sigma_{w_0}) - \partial_{i_1}(\ell(x)) \cdot \partial_{i_2} \cdots \partial_{i_\ell}(\sigma_{w_0})$

$= \sum_{i=1}^{\ell} \big(-\partial_{i_k} (s_{i_{k-1}} \cdots s_{i_1}(\ell(x)) \cdot \partial_{i_1} \cdots \hat{\partial}_{i_k} \cdots \partial_{i_\ell}(\sigma_{w_0})$

$\uparrow$ the chain rule $\ell_a - \ell_b$

The "main problem" of Schubert calculus.

Find generalized Littlewood-Richardson coeff's $c_{uv}^{w} \in \mathbb{Z}_{\geq 0}$ s.t.

$$\sigma_u \cdot \sigma_v = \sum_{w \in S_n} c_{uv}^{w} \sigma_w$$

$$X_i = \sum_{j=1}^{n} T_{ij} \in FK_n$$

<u>Lemma</u>: $X_1, \ldots, X_n$ commute with each other

$f(X_1, \ldots, X_n)$ is well defined in C_n for any polynomial f

X_i acts on C_n as the operator $X_i : f \mapsto x_i f$.

<u>Cor</u>: $\underset{\substack{\cap \\ FK_n}}{\mathfrak{S}_u (X_1, \ldots, X_n)} (\sigma_v) = \sum_w c_{uv}^{w} \sigma_w$

$$\sum_{\substack{\text{pipe} \\ \text{dreams}}} \left(\begin{array}{l} \text{a certain sum over} \\ \text{paths } u < \cdots < w \\ \text{in Strong Bruhat order} \end{array} \right)$$

But this is not a subtraction-free formula

<u>E.x.</u> $X_2 = T_{21} + T_{23} + T_{24} + \ldots$

$= -T_{12} + T_{23} + T_{24} + \ldots$

<u>Conjecture</u>: Fomin-Kirillov's conj.

$\mathfrak{S}_w (X_1, \ldots, X_n) \in FK_n$ can be written as a positive expression in T_{ij}'s, $i<j$

* <u>Known for some special cases</u>

$$\mathfrak{S}_{s_k} (X_1, \ldots, X_n) = X_1 + X_2 + \ldots + X_k$$

$$= \sum_{a \leq k < b} T_{ab} \qquad \text{(Monk's case)}$$

<u>Pieri formulas</u>

$$\mathfrak{S}_{s_{m-k+1} s_{m-k+2} \cdots s_m} = e_k (x_1, \ldots, x_m)$$

$$\mathfrak{S}_{s_{m+k-1} s_{m+k-2} \cdots s_m} = h_k (x_1, \ldots, x_m)$$

<u>Thrm</u>: In FK_n

$$e_k (X_1, \ldots, X_m) = \sum_{\substack{(1)\, i_1, \ldots, i_k \in [m] \\ (2)\, j_1, \ldots, j_k \in \{m+1, \ldots, n\} \\ (3)\, i_1, \ldots, i_k \text{ distinct} \\ (4)\, j_1 \leq j_2 \leq \ldots \leq j_k}} T_{i_1 j_1} T_{i_2 j_2} \cdots T_{i_k j_k}$$

$h_k (X_1, \ldots, X_m)$ same rule with (3) & (4) switched.

Ex. $n=4$

$e_2(X_1,X_2) = X_1 \cdot X_2$

$= (T_{12} + T_{13} + T_{14})(-T_{12} + T_{23} + T_{24})$

$= T_{12} \cdot T_{23} + T_{12}T_{24} - T_{13}T_{12} + T_{13}T_{23} + T_{13}T_{24} - T_{14}T_{12} + T_{14}T_{23} + T_{14}T_{24}$

$\to T_{23} + T_{13}$ Now all coeff's have been made positive $\to T_{24} + T_{14}$

$S_\lambda(X_1, \dots, X_m)$

If λ is a hook or hook + single box

Conjecture has been checked, but even for it is very complicated

Even proving conjecture for e.g. would be good!

LECTURE 19 10/23

Recall: • $\mathfrak{S}_w$, $w \in S_\infty$ form a linear basis of $\mathbb{C}[x_1, x_2, \dots]$
• (The cosets of) $\mathfrak{S}_w$, $w \in S_n$ form a linear basis of the <u>coinvariant algebra</u> $C_n = \mathbb{C}[x_1, \dots, x_n]/I_n$, $I_n = \langle e_1, \dots, e_n \rangle$ ← elementary symmetric polys

<u>S_n-harmonic polynomials & Dual Schubert polynomials</u>

Def: <u>D-pairing of polynomials</u> $f, g \in \mathbb{C}[x_1, \dots, x_n]$
$\langle f, g \rangle :=$ the constant term
$$f\left(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \dots, \tfrac{\partial}{\partial x_n}\right)(g(x_1, \dots, x_n))$$
Exercise: Show $\langle f, g \rangle = \langle g, f \rangle$

Ex. $\{x^a = x_1^{a_1} \cdots x_n^{a_n}\}$ is a lin. basis of $\mathbb{C}[x_1, \dots, x_n]$.
The dual basis $\left\{\frac{x_1^{a_1}}{a_1!} \cdots \frac{x_n^{a_n}}{a_n!}\right\}$

Lemma: Two basis $\{f_u\}$, $\{g_u\}$ are D-dual to each other iff
$$\boxed{\sum_u f_u(x_1, \dots, x_n)\, g_u(y_1, \dots, y_n) = e^{(x,y)}}$$
↖ could have ∞-ly many terms

Where $(x, y) = x_1 y_1 + \dots + x_n y_n$
and $e^{(x,y)} = \sum_{k \geq 0} \frac{(x,y)^k}{k!}$

Proof $\sum_{a_1 \dots a_n} x_1^{a_1} \cdots x_n^{a_n} \cdot \frac{y_1^{a_1}}{a_1!} \cdots \frac{y_n^{a_n}}{a_n!} = e^{x_1 y_1} \cdots e^{x_n y_n} = e^{(x,y)}$

Def: The space H_n of <u>S_n-Harmonic polynomials</u> is
$$H_n := \left\{f \in \mathbb{C}[x_1, \dots, x_n] \mid \forall g \in I_n,\ g\left(\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_n}\right)(f) = 0\right\}$$
Ex. H_2 $f(x_1, x_2)$
$$\begin{cases} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)(f) = 0 \\ \frac{\partial^2}{\partial x_1 \partial x_2}(f) = 0 \end{cases}$$
$x_1 - x_2$, $1 \in H_2$
One can check that in fact 1, $x_1 - x_2$ span H_n

Note: H_n closed under taking partial derivatives

Lemma: $H_n \subset \mathbb{C}[x_1, \dots, x_n]$ is D-dual to $C_n = \mathbb{C}[x_1, \dots, x_n]/\langle e_1, \dots, e_n \rangle$
In particular, H_n is $n!$ dimensional.

Ex. Vandermonde det
$$\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j) \in H_n$$
Also any partial derivative $\left(\frac{\partial}{\partial x_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{a_n} (\Delta_n) \in H_n$

Exercise: Prove that Δ_n together with its partial derivatives span all of H_n

Def: Dual Schubert polynomials $\mathfrak{D}_w$

For $w \in S_n$, we define

$$\mathfrak{D}_w(y_1,\dots,y_n) = \frac{1}{\ell(w)!}\sum_{P:(id \lessdot w^{(1)} \lessdot \cdots \lessdot w^{(\ell)} = w)} wt(P)$$

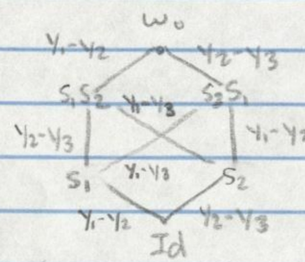
a saturated chain from id to w in strong Bruhat order

where

$$wt(P) = \prod_{i=1}^{\ell} wt(w^{(i-1)} \lessdot w^{(i)})$$

$$wt(u \lessdot ut_{ij}) = y_i - y_j$$

Ex. $n=3$



$\mathfrak{D}_{Id} = 1 \quad \mathfrak{D}_{s_1} = y_1 - y_2 \quad \mathfrak{D}_{s_2} = y_2 - y_3$

$$\mathfrak{D}_{s_1s_2} = \frac{1}{2}\left((y_1-y_2)(y_2-y_3) + (y_2-y_3)(y_1-y_3)\right)$$
$$= \frac{1}{2}(y_2-y_3)(2y_1-y_2-y_3)$$

$$\mathfrak{D}_{s_2s_1} = \frac{1}{2}(y_1-y_2)(y_1+y_2-2y_3)$$

$$\mathfrak{D}_{w_0} = \frac{1}{6}(\dots) = \frac{1}{2}(y_1-y_2)(y_1-y_3)(y_2-y_3)$$

Thrm: $\mathfrak{D}_w$, $w \in S_n$ is the linear basis of S_n which is D-dual to the basis $\{\mathfrak{S}_w \mid w \in S_n\}$ of the coinvarient alg. C_n

Thrm': $\{\mathfrak{D}_w, w \in S_\infty\}$ is the D-dual basis to $\{\mathfrak{S}_w, w \in S_\infty\}$ in $\mathbb{C}[x_1, x_2, \dots]$

(These two versions imply each other with just a little extra proving)

Proof of Thrm': Need to check the identity

$$\sum_{w \in S_\infty} \mathfrak{S}_w(x)\mathfrak{D}_w(y) = e^{(x,y)}$$

Fix ℓ. ETS $\displaystyle\sum_{\substack{w \in S_\infty \\ \ell(w)=\ell}} \mathfrak{S}_w(x)\mathfrak{D}_w(y) = \frac{1}{\ell!}(x,y)^\ell$

$$\frac{1}{\ell!}\sum_{P(id \lessdot \dots \lessdot w)} wt(P)$$

$$\Rightarrow \text{NTS} \sum_{\substack{w \in S_\infty \\ \ell(w)=\ell}} \sum_{P:(id \lessdot \dots \lessdot w)} wt(P)\mathfrak{S}_w(x_1,\dots,x_n) = (x,y)^\ell$$

$$(x_1y_1 + \dots + x_ny_n)^\ell \mathfrak{S}_{id}$$

Monks Formula:

$$(y_1x_1 + \dots + y_nx_n)\mathfrak{S}_u(x_1,\dots,x_n) = \sum_{u \lessdot ut_{ij}} (y_i - y_j)\mathfrak{S}_{ut_{ij}}$$

Thrm: $w_0 \in S_n$ longest perm

$$\mathfrak{D}_{w_0}(y_1,\dots,y_n) = \frac{1}{1!2!\cdots(n-1)!}\prod_{i<j}(y_i - y_j)$$

(Stembridge formula)

Cor: $\sum_{P:(id < \cdots < w_0)} wt(P) = \frac{\binom{n}{2}!}{1!\,2!\cdots(n-1)!} \prod_{i<j} (y_i - y_j)$

Thrm: $w \in S_n$

$\mathfrak{D}_w(y_1, \ldots, y_n) = \mathfrak{S}_{w_0 w}\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)\mathfrak{D}_{w_0}$

LECTURE 22 10/30

Symmetric Functions

$\Lambda_n = \mathbb{C}[x_1, \dots, x_n]^{S_n}$ ring of symmetric polynomials

$\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$ $\quad \Lambda_n^k$ space of homogeneous symetric polys in $x_1, \dots, x_n$ of degree k

$$\Lambda_0^k \leftarrow \Lambda_1^k \leftarrow \dots \leftarrow \Lambda_{n-1}^k \leftarrow \Lambda_n^k \leftarrow \dots$$

$$f(x_1, \dots, x_{n-1}, 0) \leftarrow f(x_1, \dots, x_n)$$

$\Lambda^k := \varprojlim \Lambda_n^k$ (projective limit)

Def: $\Lambda := \bigoplus_{k \geq 0} \Lambda^k$ the ring of symmetric functions

Ex. $x_1 + x_2 + \dots \in \Lambda$

$\sum_{i,k \geq} x_i^k \notin \Lambda$, $\prod_{i \geq 1} x_i \notin \Lambda$ b/c degrees not bounded

- monomial basis of Λ: $\{m_\lambda \; \lambda = (\lambda_1, \dots, \lambda_\ell)$ partition$\}$

$m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$ + all other monomials in $x_1, x_2, \dots$ whose exponents are given by permutations of $\lambda_1, \lambda_2, \dots, \lambda_\ell$

Lemma: $\dim \Lambda^k = \#$ partitions of k

- elementary sym. function basis $\{e_\lambda \mid$ partitions $\lambda\}$

$e_k := \sum_{i_1 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ $\qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$

- complete homogen. sym. functions $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$

$h_k := \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$

Thrm: Fundamental Thrm of Symetric functions

$\{e_\lambda\}$ and $\{h_\lambda\}$ are lin bases of Λ

$\Lambda = \mathbb{C}[e_1, e_2, \dots] = \mathbb{C}[h_1, h_2, \dots]$

- Schur Sym fncns $\{S_\lambda \; \lambda$ a partition$\}$

$s_\lambda = s_\lambda(x_1, x_2, \dots) = \sum_{\substack{T \in SSYT \\ \text{of shape } \lambda}} x^{\text{weight}(T)}$

$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$

Def: Kostka numbers

$K_{\lambda\mu} := \#$SSYT of shape λ & weight μ

The Hall inner product on Λ

Def: The Hall inner product $\langle \circ , \circ \rangle$ is defined by
$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$ (h_λ & m_μ are dual bases)

Thrm: $\prod_{i,j \geq 1} \frac{1}{1-x_i y_j} \overset{RSK}{=} \sum_\lambda s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, y_2, \dots)$

$\overset{(*)}{=} \sum_\lambda m_\lambda(x_1, x_2, \dots) h_\lambda(y_1, y_2, \dots)$

Proof of $(*)$: LHS $= \prod_{i,j} (1 + x_i y_j + (x_i y_j)^2 + \dots)$
The coeff of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$ in is $h_{\lambda_1}(y) h_{\lambda_2}(y) \cdots = h_\lambda(y)$
$\Leftrightarrow$ LHS $= \sum_\lambda m_\lambda(x) h_\lambda(y)$ ☒

Lemma: Two lin. bases $\{u_\lambda\}$, $\{v_\lambda\}$ of Λ are dual (w.r.t. Hall's product)
iff $\sum_\lambda u_\lambda(x) \cdot v_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1-x_i y_j}$

Proof: $u_\lambda = \sum_\nu a_{\lambda\nu} h_\nu$
$v_\mu = \sum_\gamma b_{\mu\gamma} m_\gamma$

$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \Leftrightarrow \langle \sum_\nu a_{\lambda\nu} h_\nu, \sum_\gamma b_{\mu\gamma} m_\gamma \rangle = \delta_{\lambda\mu}$
$\Leftrightarrow \sum_\nu a_{\lambda\nu} b_{\mu\nu} = \delta_{\lambda\mu} \Leftrightarrow A \cdot B^T = I$

$\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y)$
$\hookrightarrow = \sum_\lambda (\sum_\nu a_{\lambda\nu} h_\nu(x))(\sum_\gamma b_{\lambda\gamma} m_\gamma(y))$
$\Leftrightarrow \sum_\nu a_{\lambda\nu} \cdot b_{\lambda\gamma} = \delta_{\nu\gamma} \Leftrightarrow A^T \cdot B = I$

Both cases $\Leftrightarrow B = (A^{-1})^T$ ☒

Skew Schur functions

λ/μ skew Young diagrams
$s_{\lambda/\mu} := \sum_{\substack{T \text{ SSYT} \\ \text{of shape } \lambda/\mu}} x^{\text{weight}(T)}$

Thrm: $\langle s_\lambda, f \cdot s_\mu \rangle = \langle s_{\lambda/\mu}, f \rangle \quad \forall f \in \Lambda$

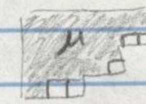
Ex. $f = h_1 = s_\square = x_1 + x_2 + \dots$
RHS $= \begin{cases} 1 & \text{if } \lambda \gtrdot \mu \\ 0 & \text{otherwise} \end{cases}$ ($\lambda = \mu$ + a single box)

$h_1 \cdot s_\mu = \sum_{\lambda \gtrdot \mu} s_\lambda$

More generally this is a special case of

Pieri Formula for Schur

$$h_k \cdot s_\mu = \sum_{\substack{\lambda : \lambda/\mu \text{ is} \\ \text{a horizontal} \\ k \text{ strip}}} s_\lambda$$



Def: Little Richardson coefficients

def 1 $c^\lambda_{\mu\nu} := \langle s_\lambda , s_\mu s_\nu \rangle = \langle s_{\lambda/\mu} , s_\nu \rangle$

def 2 $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$

def 3 $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$

LR-rule

$c^\lambda_{\mu\nu} = \#$ LR-tableau of shape λ/μ & weight ν

Little-Richardson rule is closely related to crystal bases!